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A COMBINATORIAL PROBLEM ASSOCIATED WITH A FAMILY OF COMBINATION LOCKS

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This paper discusses an interesting combinatorial problem which arises in the analysis of a family of electrical combination locks. These devices, unlike mechanical combination locks, have the sequential codes (positions of the tumblers) introduced by switch closures. Obviously a great many restrictions could be imposed on the choice of codes, their sequence of insertion etc., which would lead to functionally different types of locks. However, all of these electronic combination locks have some features in common. The operator is presented with n switches (the dial and tumbler), each of which may be set to an "on" or "off" position. The arrangement of settings of these switches constitutes a code for insertion into the lock, corresponding to the rotation of the tumbler to one of the code positions in the mechanical analog. The direction of rotation of the tumbler is reversed when the code position is reached to "enter" the code into the mechanical lock. By direct analogy, once the desired code is set into the n switches by the operator, he presses an entry button to cause the code to be entered in the lock.

The family of locks which are the origin of the problems considered in this paper have some special restrictions imposed on the code sequences allowed to the operator. The n switches are independent, i.e., they may be closed individually or in any combination whatsoever. If a switch is on when an entry is made, it is removed from further consideration. It may be useful to think of this switch being deactivated by a latching mechanism which was actuated by the initial closure of the switch and the entry button. At any rate it is not available to the operator for further code construction. There is no meaning to an ordering of the switches which are closed to form a code at the time of entry. Thus AB means the same thing as BA if the product symbol pairing is used to represent an entry code of switch A and switch B closed. The sequencing of discrete entries constitutes the "combination" of the lock. For an example A—BC would symbolize the entry of switch A in the on position followed by the entry of switches B and C in the on position. This would be a specific combination, and would be recognized as different from AB—C, etc. The locks considered here differ from their mechanical analogs in only one particular: the concept of limited try. After the operator has entered some combination, sequence of codes, he presses a "test" button. If the correct combination has been entered the lock is operated, or opens. If the combination is an incorrect one, however, the lock is disabled, temporarily or permanently, so that further combinations may not be tried.

The obvious question, in view of the foregoing description of the operation of the lock, is the security which a particular lock affords, i.e., a lock with n switches. Since the system is designed to allow only a single combination trial, the security is the probability of an unauthorized person's finding the correct code sequence by accident on the first attempt. This is $1/P_n$, where P_n is the total number of combinations for an n switch lock. P_n and several of its interesting relations are developed in the following sections.

It is possible to determine P_n directly by enumeration of the possible groupings for small n; however, this quickly proves to be an impractical technique for locks which are still of feasible size; $n=10$ for an example. The first few values of P_n are:

$$P_1 = 1 \qquad P_2 = 5 \qquad P_3 = 25.$$

The actual enumeration of the combinations for $n=1$, 2, and 3 are as follows, with the null combination, corresponding to no switches being closed, being included for logical consistency. This arrangement is of no practical interest, since it corresponds to the lock being ready to open all the time and is not considered to be an acceptable combination code for a lock. The entries in the table are grouped into columns according to the number of distinct entries involved. Thus the simultaneous entry of any number of switches is interpreted as a single distinct code entry for the purposes of this classification.

	0	1	2	3
P_1	—	$-a-$		
P_2	—	$-a-$	$-a-b-$	
		$-b-$	$-b-a-$	
		$-ab-$		
P_3	—	$-a-$	$-a-b-$	$-a-b-c-$
		$-b-$	$-b-a-$	$-b-c-a-$
		$-c-$	$-b-c-$	$-c-a-b-$
		$-ab-$	$-c-b-$	$-a-c-b$
		$-bc-$	$-a-c-$	$-c-b-a-$
		$-ac-$	$-c-a-$	$-b-a-c-$
		$-abc-$	$-a-bc-$	
			$-bc-a-$	
			$-ab-c-$	
			$-c-ab-$	
			$-ac-b-$	
			$-b-ac-$	

The above display contains the key to the solution of the problem. To form any combinatory element for the columns of P_n it suffices to note that only one new symbol is being introduced, x_n, $x=a, b, c, \cdots$. If the column elements being investigated have j distinct code entries, then it is obvious that only the columns devoted to j and $j-1$ code entries in the preceding classification of P_{n-1} can affect these elements. For an example, the new symbol, $-x_n-$, may be introduced in place of any occurrence of $-$ in the $j-1$ entry elements for P_{n-1} to produce combinations of j distinct entries. If $G(n, j)$ is defined to be the number of elements for j distinct closures under P_n, then the foregoing rule may be written symbolically as $jG(n-1, j-1)$. If one now considers the elements for P_{n-1} which involved j distinct entries, it is obvious that the introduction of the new symbol must be made in such a manner that no new entries are introduced. This can be accomplished either by including the new symbol with any product symbol grouping, code, in these combinatory elements, i.e., enter the associated switch in combination with one of the other distinct code entries, or else by not using it at all. The number of such options is expressed by $(j+1)G(n-1, j)$. These are the only ways in which combinations of j distinct entries can be generated using n symbols, given the combinations for $(n-1)$ symbols classified according to the number of entries. This result is expressed by the partial difference equation

$$G(n, j) = jG(n-1, j-1) + (j+1)G(n-1, j). \tag{1}$$

This equation is descriptive of the system being studied, and its solution and an investigation of the properties of these solutions is the object of this paper. The number P_n which is desired as a solution for the combination lock problem is given by

$$P_n = \sum_{j=1}^{n} G(n, j). \tag{2}$$

A tabular display of the numbers generated by (1) is helpful in visualizing some of the relations to be developed in the following analysis:

n/j	0	1	2	3	4	5
0	1					
1	1	1				
2	1	3	2			
3	1	7	12	6		
4	1	15	50	60	24	
5	1	31	180	390	360	120

Equation (1) can be reduced to a simpler form by the change of variable:

$$G(n, j) = j!G'(n, j). \tag{3}$$

The substitution of (3) into (1) yields the following linear partial difference equation:

$$G'(n, j) = G'(n-1, j-1) + (j+1)G'(n-1, j). \tag{4}$$

To allow symbolic manipulation this is best expressed in terms of the partial displacement operators, $\mathbf{E}_n$ and $\mathbf{E}_j$, defined by the relation $\mathbf{E}_nG(n, j) = G(n+1, j)$, [1]. Equation (4) may then be written as:

$$\left(\mathbf{E}_n\mathbf{E}_j - 1 - (j+2)\mathbf{E}_j\right)G'(n, j) = 0. \tag{5}$$

This type of linear partial difference equation, i.e., one in which one of the variables does not appear explicitly, may be treated by Boole's method. Consider the operator $\mathbf{E}_n$ to be a constant k and solve the resulting linear difference equation in the single variable j. Equation (5) now assumes the form

$$(k - j - 2)\mathbf{E}_j G(n, j) - G'(n, j) = 0. \tag{6}$$

If the variable change

$$u(n, j) = v(j)G'(n, j) \tag{7}$$

is made, where $v(j)$ is defined to be

$$v(j) = (k-1)(k-2)\cdots(k-j-1), \tag{8}$$

(6) may be reduced to the simple system

$$u(n, j+1) - u(n, j) = 0. \tag{9}$$

Equation (9) has as a solution

$$u(n, j) = c \tag{10}$$

which may be rewritten in the following form using (8):

$$[k-1][k-2]\cdots[k-(j+1)]G'(n, j) = c. \tag{11}$$

This may then be expanded into the following operator ($\mathbf{E}_n$) equation:

$$\sum_{i=0}^{j} k^i \mathbf{S}_{j+1}^{i+1} G'(n, j) = c, \tag{12}$$

where $\mathbf{S}_{j+1}^{i+1}$ is a Stirling number of the first kind.

This equation, (12), may be treated by the method of characteristic equations. The characteristic equation associated with (12) is

$$\sum_{i=0}^{j} \mathbf{S}_{j+1}^{i+1} r^{j-1} = 0. \tag{13}$$

The roots of this equation are $1, 2, 3, \cdots, j+1$, so that the general solution of (12) is of the form

$$G'(n, j) = c_0(j)(j+1)^n + c_1(j)j^n + \cdots + c_{j-1}(j)2^n + c_j, \tag{14}$$

where the $c_i(j)$ are arbitrary, and as yet undetermined, functions of j.

A single value of the function P_n will involve $n(n+1)/2$ of the functions, $c_i(j)$, or rather functional values. In order to determine these functions it is necessary to refer to the defining relationship (4). Examination of (4) allows the following boundary conditions to be stated:

$$\begin{aligned} G'(n, j) &= 0 \qquad \text{if } j > n \\ G'(n, j) &= 1 \qquad \text{if } j = n. \end{aligned} \tag{15}$$

If these restrictions are imposed on (14) the following family of simultaneous equations is obtained:

$$\left.\begin{aligned} &\sum_{i=0}^{j} c_i(j)(j+1-i)^n = 1 \\ &\sum_{i=0}^{j} c_i(j)(j+1-i)^{n-k} = 0 \qquad \text{for } 1 \leqq k \leqq n \end{aligned}\right\} \tag{16}$$

The determinant of this system is

$$(17)\qquad \begin{vmatrix} (j+1)^n & j^n & \cdots 2^n & 1 \\ (j+1)^{n-1} & j^{n-1} & \cdots 2^{n-1} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ (j+1) & j & \cdots 2 & 1 \\ 1 & 1 & \cdots 1 & 1 \end{vmatrix} = D$$

which is Vandermonde's determinant. n appears as a dummy variable in (17) as may be seen by examining the restraints of (15) and the equations (16) from which D is derived. The system is defined only for $0 \leqq j \leqq n$, i.e., in solving for the $c_i(j)$ only $(j+1)$ equations of the form given by (14) are considered of degrees $(j+1)$, (j), $\cdots$, (2),(1). Thus (17) is a square $(j+1)\times(j+1)$ determinant.

D is given by

$$(18)\qquad D = j!(j-1)!\cdots 2!\,1!$$

Since the constant column to be introduced into D for the solution for the $c_i(j)$ is a unique one, i.e., a one in the highest order position and zero in all other entries, a very simple solution is possible, based on an expansion of D by minors along the upper row. If one denotes the numerator matrices by N_i, then they may be written in the following form:

$$(19)\qquad N_i = \frac{(j-1)!(j-2)!\cdots 2!1!}{i!(j-1-i)!}(-1)^i,$$

where i obviously ranges from 0 to j. The $c_i(j)$ may now be determined:

$$(20)\qquad c_i(j) = \frac{N_i}{D} = (-1)^i\binom{j}{i}\frac{1}{j!}\,.$$

Thus the arbitrary functions $c_i(j)$ are determined and the solution may be written for (14) with the $c_i(j)$ replaced by the appropriate functions of j.

$$(21)\qquad G'(n,j) = \sum_{i=0}^{j}(-1)^i\binom{j}{i}\frac{1}{j!}(j+1-i)^n$$

or

$$(22)\qquad G(n,j) = \sum_{i=0}^{j}(-1)^i\binom{j}{i}(j+1-i)^n.$$

The form assumed by the $G'(n,j)$ is a well-known expression for the Stirling numbers of the second kind, [2]. The solutions (21) and (22) may be conveniently written in a simpler form by the introduction of $\mathfrak{S}_{n+1}^{j+1}$.

$$(21\text{a})\qquad G'(n,j) = \mathfrak{S}_{n+1}^{j+1}$$

$$(22\text{a})\qquad G(n,j) = j!\,\mathfrak{S}_{n+1}^{j+1}.$$

This completes the solution of the original partial difference equation (1), and incidentally of the problem from which it was derived. Equation (2) may be used with either (22) or (22a) to give as a final result

$$P_n = \sum_{j=1}^{n} \sum_{i=0}^{j} (-1)^i \binom{j}{i} (j + 1 - i)^n \tag{23}$$

or

$$P_n = \sum_{j=1}^{n} j! \mathfrak{S}_{n+1}^{j+1}. \tag{23a}$$

The generating function for the P_n is simply obtained from the latter form, (23a), and is found to be:

$$P_n = \frac{d^n}{du^n}\left(\frac{1}{2e^{-u} - 1}\right)_{u=0} - 1, \tag{24}$$

where the nth derivative of the parenthetic term is to be evaluated at $u=0$.

It is of interest to tabulate some of the values of P_n derived by use of (23a), since they are the measure of security achieved by the family of combination locks on which this problem is based.

	1	2	3	4	5	6	7	8	9	10
P_n	1	5	25	149	1,081	9,366	94,586	1,091,670	14,174,522	204,495,125

References

1. Charles Jordan, Calculus of finite differences, Chelsea, New York, 1950.
2. John Riordan, An introduction to combinatorial analysis, Wiley, New York, 1958.

INVARIANTS ASSOCIATED WITH A FAMILY OF CURVES

F. S. NOWLAN, The Late Professor Emeritus, University of Illinois and
A. A. AUCOIN, University of Houston

I. Introduction. We consider in this paper invariants associated with the polar coordinate family of curves $\rho_n = \rho_n(\theta)$, generated by transforming a base curve $\rho_0 = \rho_0(\theta)$ by the transformation

$$\rho_n = g^n \rho_0, \qquad \theta_n = \theta, \tag{1}$$

where $g = g(\theta)$ is an arbitrary function and n is any integer. For $n=0$, the transformation yields the base curve (the transformation (1) is a special case of the general transformation $w_n(u) = g^n(u)w_0(u)$, $u_n = u$, where w and u refer to any system of plane coordinates. All invariants derived are of the same form in any coordinate system, but their interpretations differ). All invariants are evaluated

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We observe that a necessary and sufficient condition that a curve $r=r(\theta)$ be a member of the family generated by (1) is that r/ρ_0 be expressible in the form g^m, where m is some definite integer.

II. Nondifferentiable Invariants. From (1) we note that the ρ_n form a geometric progression. The properties of geometric progressions yield the invariants (the integer n is not necessarily nonnegative, but in any case $-n$ is the negative of n)

$$(2)\qquad \prod_{k=-s}^{s} \rho_{n+kr} - \rho_n^{2s+1} = 0, \qquad\qquad (4)\qquad \rho_n\rho_{-n} = \rho_0^2,$$

$$(3)\qquad \rho_{n+r}\rho_{n-r} - \rho_n^2 = 0, \qquad\qquad (5)\qquad \frac{\rho_n}{\rho_{n-1}} = g.$$

Two additional results are true for numbers in geometric progression. If $(ABCD)$ denotes the cross ratio of A, B, C, D then we have $(\rho_{n_1}\rho_{n_2}\rho_{n_3}\rho_{n_4}) = (\rho_{-n_1}\rho_{-n_2}\rho_{-n_3}\rho_{-n_4})$ and since $(\rho_0 - \rho_0\rho_n\rho_{-n}) = -1$, the elements ρ_0, $-\rho_0$, ρ_n, ρ_{-n} form a harmonic set.

III. Differentiable Invariants. All derivatives will be with respect to θ. We use the notation

$$\rho_n'' = \frac{d^2\rho_n}{d\theta^2}, \qquad \rho_n^{(s)} = \frac{d^s\rho_n}{d\theta^s},$$

etc. We further assume the existence of all derivatives to which we are led.

We now consider three types of invariants that concern derivatives. The results are contained in three theorems. Geometric applications of these theorems will be made later.

THEOREM 1. *If the d_i are real numbers and the n_i are any integers, then*

$$(6)\qquad \sum_{i=1}^{s} d_i \frac{\rho_{n_i}'}{\rho_{n_i}} - \frac{g'}{g}\sum_{i=1}^{s} d_i n_i = \frac{\rho_0'}{\rho_0}\sum_{i=1}^{s} d_i.$$

Proof. From (1)

$$(7)\qquad \prod_{i=1}^{s} \rho_{n_i}^{d_i} = g^{\lambda}\rho_0^{\mu},$$

where $\lambda = \sum_{i=1}^{s} d_i n_i$, $\mu = \sum_{i=1}^{s} d_i$. The result now follows by differentiating the logarithm of (7).

If in (6) we choose $s=2$, $n_1=n$, $n_2=n-1$, $d_1=-d_2=1$ then

$$(8)\qquad \frac{\rho_n'}{\rho_n} - \frac{\rho_{n-1}'}{\rho_{n-1}} = \frac{g'}{g}.$$

From (8) it follows that the ρ_n'/ρ_n form an arithmetic progression. The properties of arithmetic progressions yield, for every n

$$\frac{\rho_{n+r}'}{\rho_{n+r}} + \frac{\rho_{n-r}'}{\rho_{n-r}} - 2\frac{\rho_n'}{\rho_n} = 0, \tag{9}$$

$$\frac{\rho_n'}{\rho_n} + \frac{\rho_{-n}'}{\rho_{-n}} = 2\frac{\rho_0'}{\rho_0}\,. \tag{10}$$

We note that (8), (9), and (10) may also be obtained by differentiating the logarithm of (5), (3), and (4) respectively. (We note, incidentally, that the derivatives of the logarithm of the terms of a geometric progression, with respect to the common ratio, are in arithmetic progression.)

THEOREM 2. *If the d_i are any real numbers and the n_i are any integers, then*

$$\sum_{i=1}^{s} d_i\left[\frac{\rho_{n_i}''}{\rho_{n_i}} - \left(\frac{\rho_{n_i}'}{\rho_{n_i}}\right)^2\right] - \left[\frac{g''}{g} - \left(\frac{g'}{g}\right)^2\right]\sum_{i=1}^{s} d_i n_i = \left[\frac{\rho_0''}{\rho_0} - \left(\frac{\rho_0'}{\rho_0}\right)^2\right]\sum_{i=1}^{s} d_i. \tag{11}$$

Proof. Since

$$\frac{d}{d\theta}\frac{\rho_n'}{\rho_n} = \frac{\rho_n''}{\rho_n} - \left(\frac{\rho_n'}{\rho_n}\right)^2, \tag{12}$$

the theorem follows by differentiating (6).

Further general theorems can be obtained by successively differentiating (6). For example, the next differentiation would involve terms of the form

$$\frac{\rho_n'''}{\rho_n} - 3\frac{\rho_n''}{\rho_n}\frac{\rho_n'}{\rho_n} + 2\left(\frac{\rho_n'}{\rho_n}\right)^3.$$

To prove Theorem 3 we need two lemmas.

LEMMA 1. *Let $P_s(x) = a_s x^s + a_{s-1}x^{s-1} + \cdots + a_0$ be a polynomial of degree s in x. Then*

$$\sum_{k=0}^{s} (-1)^k \binom{s}{k} P_r(k) = \begin{cases} (-1)^s s!\, a_s & \text{if } r = s \\ 0 & \text{if } r < s. \end{cases}$$

Proof. See F. S. Nowlan, "The Evaluation of Summations with Binomial Coefficients," this MAGAZINE, 34 (1961) 161–163. See also, Solution No. 3844, *Amer. Math. Monthly*, 46 (1939) 658–660.

LEMMA 2. *For every value of n*

$$\frac{\rho_n^{(s)}}{\rho_n} = n^s\left(\frac{g'}{g}\right)^s + P_{s-1}(n).$$

Proof. The proof is by induction. The step from k to $k+1$ is obtained by differentiating the expression for k.

We are concerned in Theorem 3 with sums of terms, with binomial coefficients, of the form

$$\prod_{i=1}^{m}\left[\frac{\overset{(\alpha_i)}{\rho_{n_i+p_ik}}}{\rho_{n_i+p_ik}}\right]^{\beta_i},$$

where α_i, β_i, n_i, p_i, k are integers, α_i, β_i being positive. The weight of a summation is defined to be $\sum_{i=1}^{m}\alpha_i\beta_i$. It will be shown that all summations of the same weight have the same value, independent of n_i. There are, for example, 11 summations of weight 4 given by

α_1	β_1	α_2	β_2	α_3	β_3	α_4	β_4
4	1	0	0	0	0	0	0
3	1	1	1	0	0	0	0
2	1	1	1	1	1	0	0
2	1	2	1	0	0	0	0
2	1	1	2	0	0	0	0
2	2	0	0	0	0	0	0
1	1	1	1	1	1	1	1
1	2	1	1	1	1	0	0
1	3	1	1	0	0	0	0
1	2	1	2	0	0	0	0
1	4	0	0	0	0	0	0

THEOREM 3. *If* $\sum_{i=1}^{m}\alpha_i\beta_i=r$, *then*

$$\sum_{k=0}^{s}(-1)^k\binom{s}{k}\prod_{i=1}^{m}\left[\frac{\overset{(\alpha_i)}{\rho_{n_i+p_ik}}}{\rho_{n_i+p_ik}}\right]^{\beta_i}=\begin{cases}(-1)^s s!\left(\dfrac{g'}{g}\right)^s\displaystyle\prod_{i=1}^{m}p_i^{\alpha_i\beta_i} & \text{if } r=s\\ 0 & \text{if } r<s.\end{cases}$$

Proof. From Lemma 2

$$\begin{aligned}\frac{\overset{(\alpha_i)}{\rho_{n_i+p_ik}}}{\rho_{n_i+p_ik}}&=(n_i+p_ik)^{\alpha_i}\left(\frac{g'}{g}\right)^{\alpha_i}+P_{\alpha_i-1}(n_i+p_ik)\\&=k^{\alpha_i}\left(\frac{g'}{g}\right)^{\alpha_i}p_i^{\alpha_i}+P_{\alpha_i-1}(k).\end{aligned}$$

Thus

$$\left[\frac{\overset{(\alpha_i)}{\rho_{n_i+p_ik}}}{\rho_{n_i+p_ik}}\right]^{\beta_i}=k^{\alpha_i\beta_i}\left(\frac{g'}{g}\right)^{\alpha_i\beta_i}p_i^{\alpha_i\beta_i}+P_{\alpha_i\beta_i-1}(k)\text{ and}$$

$$\begin{aligned}\prod_{i=1}^{m}\left[\frac{\overset{(\alpha_i)}{\rho_{n_i+p_ik}}}{\rho_{n_i+p_ik}}\right]^{\beta_i}&=\prod_{i=1}^{m}\left[k^{\alpha_i\beta_i}\left(\frac{g'}{g}\right)^{\alpha_i\beta_i}p_i^{\alpha_i\beta_i}+P_{\alpha_i\beta_i-1}(k)\right]\\&=k^r\left(\frac{g'}{g}\right)^r\prod_{i=1}^{m}p_i^{\alpha_i\beta_i}+P_{r-1}(k).\end{aligned}$$

We have now from Lemma 1

$$\sum_{k=0}^{s}(-1)^k\binom{s}{k}\prod_{i=1}^{m}\left[\frac{\rho^{(\alpha_i)}_{n_i+p_ik}}{\rho_{n_i+p_ik}}\right]^{\beta_i}$$

$$=\sum_{k=0}^{s}(-1)^k\binom{s}{k}\left[k^r\left(\frac{g'}{g}\right)^r\prod_{i=1}^{m}p_i^{\alpha_i\beta_i}+P_{r-1}(k)\right]$$

$$=\begin{cases}(-1)^s s!\left(\dfrac{g'}{g}\right)^s\displaystyle\prod_{i=1}^{m}p_i^{\alpha_i\beta_i} & \text{if } r=s\\ 0 & \text{if } r<s.\end{cases}$$

IV. Geometric Applications of Theorems 1 and 3. Let ψ_n denote the least positive angle measured from a vector through the pole to the tangent to ρ_n and ψ_g the corresponding angle for the function g. From the calculus, $\cot\psi_n=\rho_n'/\rho_n$ and $\cot\psi_g=g'/g$. It then follows from (8), (9), and (10) that

$$\cot\psi_n-\cot\psi_{n-1}=\frac{g'}{g}=\cot\psi_g, \tag{13}$$

$$\cot\psi_{n+r}+\cot\psi_{n-r}-2\cot\psi_n=0, \tag{14}$$

$$\cot\psi_n+\cot\psi_{-n}=2\,\frac{\rho_0'}{\rho_0}=2\cot\psi_0 \tag{15}$$

and the $\cot\psi_n$ form an arithmetic progression. In general, from Theorem 1

$$\sum_{i=1}^{s}d_i\cot\psi_{n_i}-\cot\psi_g\sum_{i=1}^{s}d_in_i=\cot\psi_0\sum_{i=1}^{s}d_i. \tag{16}$$

From (16) we have that (13), (14), and (15) follow as special cases.

Infinitely many relations involving ψ_n follow from Theorem 3. For these we have only to choose $\alpha_i=1$. For example, if we choose $s=4$, $m=3$, $n_1=n$, $n_2=n-3$, $n_3=n+4$, $p_1=2$, $p_2=4$, $p_3=-5$, $\alpha_1=\alpha_2=\alpha_3=1$, $\beta_1=\beta_3=1$, $\beta_2=2$ we have for every n

$$\cot\psi_n\cot^2\psi_{n-3}\cot\psi_{n+4}-4\cot\psi_{n+2}\cot^2\psi_{n+1}\cot\psi_{n-1}$$
$$+6\cot\psi_{n+4}\cot^2\psi_{n+5}\cot\psi_{n-6}-4\cot\psi_{n+6}\cot^2\psi_{n+9}\cot\psi_{n-11}$$
$$+\cot\psi_{n+8}\cot^2\psi_{n+13}\cot\psi_{n-16}=-3840\left(\frac{g'}{g}\right)^4,$$

while if we choose $s=4$, $m=2$, $n_1=n$, $n_2=n-3$, $p_1=2$, $p_2=4$, $\alpha_1=\alpha_2=1$, $\beta_1=1$, $\beta_2=2$,

$$\cot\psi_n\cot^2\psi_{n-3}-4\cot\psi_{n+2}\cot^2\psi_{n+1}+6\cot\psi_{n+4}\cot^2\psi_{n+5}$$
$$-4\cot\psi_{n+6}\cot^2\psi_{n+9}+\cot\psi_{n+8}\cot^2\psi_{n+13}=0.$$

V. Slope. If we let m_n be the slope of the curve ρ_n, then

$$m_n = \tan(\theta + \psi_n) = \frac{\tan\theta + \tan\psi_n}{1 - \tan\theta\tan\psi_n} = \frac{1 + \cot\psi_n \tan\theta}{\cot\psi_n - \tan\theta}. \tag{17}$$

Since

$$\cot\psi_n = \frac{\rho_n'}{\rho_n} = n\frac{g'}{g} + \frac{\rho_0'}{\rho_0} = n\frac{g'}{g} + \cot\psi_0$$

we have for all n

$$m_n = \frac{1 + \tan\theta\cot\psi_0 + n\dfrac{g'}{g}\tan\theta}{\cot\psi_0 - \tan\theta + n\dfrac{g'}{g}}.$$

From (17), for every n,

$$\cot\psi_n = \tan\theta + \frac{\sec^2\theta}{m_n - \tan\theta}.$$

Then from (13), (14), and (15) we have

$$\frac{1}{m_n - \tan\theta} - \frac{1}{m_{n-1} - \tan\theta} = \frac{g'}{g}\cos^2\theta,$$

$$\frac{1}{m_{n+r} - \tan\theta} + \frac{1}{m_{n-r} - \tan\theta} - \frac{2}{m_n - \tan\theta} = 0,$$

$$\frac{1}{m_n - \tan\theta} + \frac{1}{m_{-n} - \tan\theta} = \frac{2}{m_0 - \tan\theta}.$$

We note that the terms $1/(m_n - \tan\theta)$ form an arithmetic progression.

VI. Curvature. From the polar coordinate transformation $x = \rho_n\cos\theta$, $y = \rho_n \sin\theta$ we have

$$y' = \frac{\rho_n'\sin\theta + \rho_n\cos\theta}{\rho_n'\cos\theta - \rho_n\sin\theta}, \qquad y'' = \frac{2\rho_n'^2 - \rho_n''\rho_n + \rho_n^2}{(\rho_n'\cos\theta - \rho_n\sin\theta)^3}.$$

If K_n is the curvature of ρ_n then

$$K_n[1 + y'^2]^{3/2} = y''$$

and so

$$\rho_n K_n\left[1 + \left(\frac{\rho_n'}{\rho_n}\right)^2\right]^{3/2} - \left[1 + \left(\frac{\rho_n'}{\rho_n}\right)^2\right] = -\left[\frac{\rho_n''}{\rho_n} - \left(\frac{\rho_n'}{\rho_n}\right)^2\right].$$

Since $\rho_n'/\rho_n = \cot\psi_n$, and from (12)

$$\frac{\rho_n''}{\rho_n} - \left(\frac{\rho_n'}{\rho_n}\right)^2 = \frac{d}{d\theta}\cot\psi_n$$

$$(\rho_n K_n \csc\psi_n - 1)\csc^2\psi_n = -\left[\frac{\rho_n''}{\rho_n} - \left(\frac{\rho_n'}{\rho_n}\right)^2\right] = -\frac{d}{d\theta}\cot\psi_n. \tag{18}$$

From (13), for every n,

$$\begin{aligned}(\rho_n K_n \csc\psi_n - 1)\csc^2\psi_n - (\rho_{n-1}K_{n-1}\csc\psi_{n-1} - 1)\csc^2\psi_{n-1} \\ = -\frac{d}{d\theta}(\cot\psi_n - \cot\psi_{n-1}) \\ = -\frac{d}{d\theta}\frac{g'}{g} = \left(\frac{g'}{g}\right)^2 - \frac{g''}{g}.\end{aligned} \tag{19}$$

Thus the terms $(\rho_n K_n \csc\psi_n - 1)\csc^2\psi_n$ form an arithmetic progression. The properties of arithmetic progressions give

$$\begin{aligned}(\rho_{n+r}K_{n+r}\csc\psi_{n+r} - 1)\csc^2\psi_{n+r} + (\rho_{n-r}K_{n-r}\csc\psi_{n-r} - 1)\csc^2\psi_{n-r} \\ - 2(\rho_n K_n \csc\psi_n - 1)\csc^2\psi_n = 0,\end{aligned} \tag{20}$$

and

$$\begin{aligned}(\rho_n K_n \csc\psi_n - 1)\csc^2\psi_n + (\rho_{-n}K_{-n}\csc\psi_{-n} - 1)\csc^2\psi_{-n} \\ = 2(\rho_0 K_0 \csc\psi_0 - 1)\csc^2\psi_0.\end{aligned} \tag{21}$$

From (18) and Theorem 2 it follows, in general, that

$$\begin{aligned}\sum_{i=1}^{s} d_i(\rho_{n_i}K_{n_i}\csc\psi_{n_i} - 1)\csc^2\psi_{n_i} - (gK_g\csc\psi_g - 1)\csc^2\psi_g\sum_{i=1}^{s} d_i n_i \\ = (\rho_0 K_0 \csc\psi_0 - 1)\csc^2\psi_0\sum_{i=1}^{s} d_i.\end{aligned} \tag{22}$$

Special cases of (22) yield (19), (20), and (21).

SOME NONLINEAR DIFFERENTIAL EQUATIONS SATISFIED BY THE JACOBIAN ELLIPTIC FUNCTIONS

A. C. SOUDACK, University of British Columbia

Most of the literature on the Jacobian elliptic functions does not discuss, as an end, the use of these functions as solutions to a class of nonlinear differential equations. The purpose of this paper is to develop the various elliptic functions and to discuss a class of nonlinear differential equations satisfied by these functions.

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$$\begin{aligned}(\rho_{n+r}K_{n+r}\csc\psi_{n+r} - 1)\csc^2\psi_{n+r} &+ (\rho_{n-r}K_{n-r}\csc\psi_{n-r} - 1)\csc^2\psi_{n-r} \\ &- 2(\rho_n K_n\csc\psi_n - 1)\csc^2\psi_n = 0,\end{aligned} \tag{20}$$

and

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From (18) and Theorem 2 it follows, in general, that

$$\begin{aligned}\sum_{i=1}^{s} d_i(\rho_{n_i}K_{n_i}\csc\psi_{n_i} - 1)\csc^2\psi_{n_i} &- (gK_g\csc\psi_g - 1)\csc^2\psi_g\sum_{i=1}^{s} d_i n_i \\ &= (\rho_0 K_0\csc\psi_0 - 1)\csc^2\psi_0\sum_{i=1}^{s} d_i.\end{aligned} \tag{22}$$

Special cases of (22) yield (19), (20), and (21).

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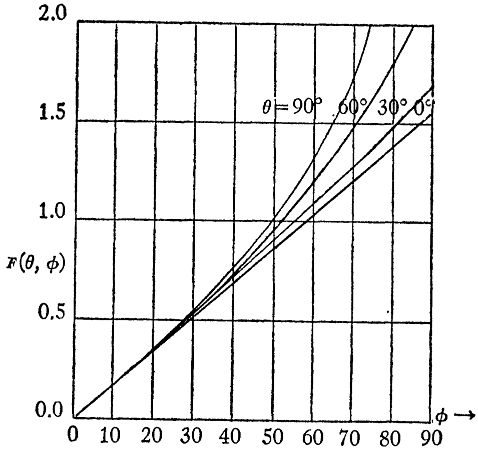

FIG. 1. The elliptic integral $F(\theta, \phi)$, $k = \sin \theta$.

1. The Jacobian Elliptic Functions. The starting point of our development will be the elliptic integral of the first kind, which we obtain in solving the equation of motion for the undamped pendulum.

This integral is

$$\omega t = u = F(k, \phi) = \int_0^{\phi} \frac{d\lambda}{\sqrt{\{1 - k^2 \sin^2 \lambda\}}} \qquad 0 \leqq \phi \leqq \frac{\pi}{2} \tag{1}$$

and is plotted for various values of "k" in Figure 1. This integral and a table of trigonometric functions are all that are needed to obtain the Jacobian elliptic

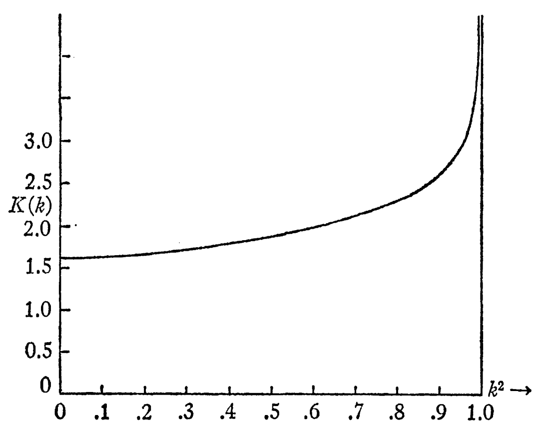

FIG. 2. $K(k)$ vs k^2.

functions. The technique for doing this is described in [1]. The quarter period of oscillation is given by $K(k)$, the complete elliptic integral of the first kind, where $K(k) = F(k, \frac{1}{2}\pi)$. A plot of $K(k)$ vs "k^2" is shown in Figure 2, from which we obtain $K(0) = \frac{1}{2}\pi$ and $K(1) = \infty$. Since we know that an undamped pendulum has an oscillatory motion for $\theta_m < \pi$, it is more instructive to deal with the function $\phi(t)$, rather than with the integral, which expresses the motion as a function $t(\phi)$.

To this end, let $x = \sin\phi$. Then $0 \leqq \phi \leqq \pi/2$ leads to $0 \leqq x \leqq 1$.

Equation (1) then becomes

$$\omega t = u = F(k, \sin^{-1} x) = \int_0^x \frac{d\mu}{\sqrt{\{(1-\mu^2)(1-k^2\mu^2)\}}}. \tag{2}$$

Jacobi introduced the notation $\phi = \text{amplitude } u \equiv \text{am } u$, $x = \text{Sin }\phi = \text{Sin (am } u)$ $= \text{Sn}(k, u) \equiv$ Jacobian elliptic sine of u, with modulus "k". For convenience, $\text{Sn}(k, u)$ will be abbreviated to Sn u, or more simply, Sn.

Equation (2) can therefore be written

$$u = \text{Sn}^{-1}x = \int_0^x \frac{d\mu}{\sqrt{\{(1-\mu^2)(1-k^2\mu^2)\}}} \qquad 0 \leqq x \leqq 1.$$

From this expression we see that $0 \leqq \text{Sn} \leqq 1$ for $0 \leqq F(k, \phi) \leqq K(k)$ i.e., Sn has a quarter period equal to $K(k)$. Since "k" is a function of the initial conditions, for example, $k = \text{Sin } \frac{1}{2}\theta_m$ for the pendulum, the period of oscillation, $4K$, is a function of the initial conditions. Since this dependence of the frequency on the initial amplitude is a property of second order nonlinear differential equations, we might suspect that the elliptic sine and its cohorts might be exact solutions to other nonlinear differential equations. We may take a step further and suggest that the elliptic functions might be good *approximations* to the solutions of other nonlinear differential equations.

It is instructive to look at the limiting cases of Sn u, namely where $k = 0$ and $k = 1$.

For $k = 0$

$$\text{Sn}^{-1}x = \int_0^x \frac{d\mu}{\sqrt{(1-\mu^2)}} = \text{Sin}^{-1} x \qquad \therefore \text{Sn}(0, u) = \text{Sin } u.$$

For $k = 1$

$$\text{Sn}^{-1}x = \int_0^x \frac{d\mu}{1-\mu^2} = \text{Tanh}^{-1} x \qquad \therefore \text{Sn}(1, u) = \text{Tanh } u.$$

The elliptic sine and, as we shall later see, the rest of the elliptic functions, can be thought of as intermediate functions between the trigonometric and hyperbolic functions. For this reason, one might suspect that these functions could be versatile approximants.

To get the Jacobian Elliptic Cosine, consider the substitution $x = \text{Cos }\phi$ $= \text{Cos(am } u) = \text{Cn}(k, u)$. Equation (1) then becomes

$$u = \text{Cn}^{-1}x = \int_x^1 \frac{d\mu}{\sqrt{\{(1-\mu^2)(k'^2+k^2\mu^2)\}}} \qquad 0 \leqq x \leqq 1,$$

where $k'^2+k^2=1$. The Cn also has a quarter period equal to $K(k)$. For $k=0$, $k'^2=1$, and

$$u = \mathrm{Cn}^{-1}x = \int_x^1 \frac{d\mu}{\sqrt{(1-\mu^2)}} = \mathrm{Cos}^{-1}\, x.$$

Therefore $\mathrm{Cn}(0, u)=\mathrm{Cos}\, u$. For $k=1$, $k'^2=0$, and

$$u = \mathrm{Cn}^{-1}x = \int_x^1 \frac{d\mu}{\mu\sqrt{(1-\mu^2)}} = \mathrm{Sech}^{-1}\, x.$$

Therefore $\mathrm{Cn}(1, u)=\mathrm{Sech}\, u$. We see that Cn is also an intermediate between the trigonometric and the hyperbolic functions.

To complete the analogy with the trigonometric functions, let

$$x = \tan\phi = \tan(\mathrm{am}\, u) = \mathrm{Tn}(k, u) = \frac{\mathrm{Sn}(k, u)}{\mathrm{Cn}(k, u)}\cdot$$

Using the transformation, equation (1) becomes

$$u = \mathrm{Tn}^{-1}x = \int_0^x \frac{d\mu}{\sqrt{\{(1+\mu^2)(1+k'^2\mu^2)\}}} \qquad 0 \leqq x \leqq \infty.$$

Here $0\leqq \mathrm{Tn}(k, u)\leqq\infty$ and $0\leqq F(k, \phi)\leqq K(k)$. For $k=0$, $k'^2=1$ and $u=\mathrm{Tn}^{-1}\, x =\int_0^x d\mu/(1+\mu^2)=\tan^{-1}\, x$. Therefore $\mathrm{Tn}(0, u)=\mathrm{Tan}\, u$. For $k=1$, $k'^2=0$ and $u=\mathrm{Tn}^{-1}\, x=\int_0^x d\mu/\sqrt{(1+\mu^2)}=\mathrm{Sinh}^{-1}\, x$. Therefore $\mathrm{Tn}(1, u)=\mathrm{Sinh}\, u$.

Again the elliptic function is an intermediate between the trigonometric and hyperbolic functions.

The rest of the trigonometric functions have their analogies in the elliptic domain. Standard nomenclature is based on the reversal of letters, yielding

$$\text{Elliptic Secant} = \frac{1}{\mathrm{Cn}(k, u)} = \mathrm{Nc}(k, u)$$

$$\text{Elliptic Cosecant} = \frac{1}{\mathrm{Sn}(k, u)} = \mathrm{Ns}(k, u)$$

$$\text{Elliptic Cotangent} = \frac{1}{\mathrm{Tn}(k, u)} = \mathrm{Nt}(k, u).$$

Other useful relations, by direct analogy to the trigonometric functions, are $\mathrm{Sn}^2\, u+\mathrm{Cn}^2\, u=1$, $1+\mathrm{Tn}^2\, u=\mathrm{Nc}^2\, u$, and $1+\mathrm{Nt}^2\, u=\mathrm{Ns}^2\, u$. This does not complete the collection of Jacobian Elliptic Functions, as there is one which has no counterpart in the trigonometric domain. It arises in the derivatives of the other elliptic functions, and is called $\mathrm{Dn}(k, u)$. The notation is due to Gudermann, the most probable reason being as follows:

$$F(k, \phi) = \int_0^\phi \frac{d\lambda}{\sqrt{\{1-k^2\,\mathrm{Sin}^2\,\lambda\}}} = \int_0^\phi \frac{d\phi}{\Delta\phi}\cdot$$

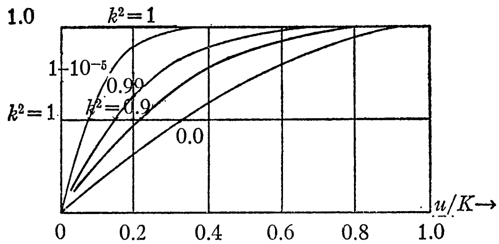

FIG. 3a. $Sn(k, u)$ vs u/K.

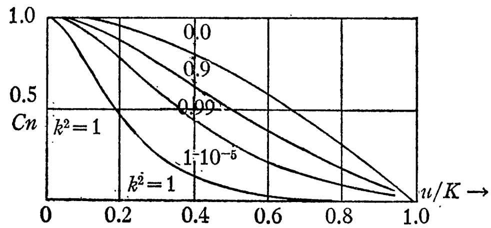

FIG. 3b. $Cn(k, u)$ vs u/K.

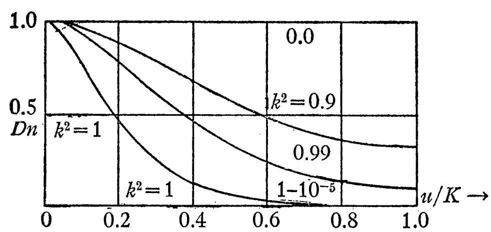

FIG. 3c. $Dn(k, u)$ vs u/K.

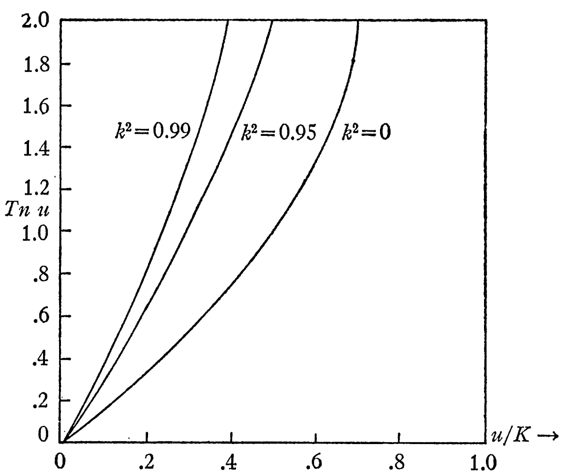

FIG. 3d. $Tn(k, u)$ vs u/K.

The abbreviation $\Delta\phi = \surd\{1-k^2 \operatorname{Sin}^2 \phi\}$ is due to Jacobi, and is, except for a variable change, equal to $\operatorname{Dn}(k, u)$. Since "Delta" starts with "D," Gudermann chose to be consistent with Sn and Cn, calling $\Delta\phi$, Dn. And so

$$\operatorname{Dn}(k, u) = \surd\{1 - k^2\operatorname{Sn}^2(k, u)\}. \tag{3}$$

Since $\operatorname{Cn} = \surd\{1-\operatorname{Sn}^2(k, u)\}$, this Dn function is "sort of a cosine," the limiting cases being $\operatorname{Dn}(0, u)=1$ and $\operatorname{Dn}(1, u) = \operatorname{Cn}(1, u) = \operatorname{Sech} u$.

The first quarter periods of the four important elliptic functions are plotted in Figure 3. Note that the abscissa is normalized to u/K, so that the quarter periods are the same for different values of "k".

We can see by inspecting the curves that Sn and Cn are *entirely different* functions, and not simply the same function displaced by a quarter period, i.e., $\operatorname{Sin}(\phi+\frac{1}{2}\pi) = \operatorname{Cos}\phi$ but $\operatorname{Sn}(k, u+K) \neq \operatorname{Cn}(k, u)$.

2. Derivatives of the Jacobian Elliptic Functions. To get the derivatives of these functions, consider

$$u = \operatorname{Sn}^{-1}x = \int_0^x \frac{d\mu}{\surd\{(1-\mu^2)(1-k^2\mu^2)\}}, \qquad x = \operatorname{Sn}(k, u).$$

Then

$$\frac{du}{dx} = \frac{1}{\surd\{(1-x^2)(1-k^2x^2)\}}$$

$$\frac{dx}{du} = \frac{d\operatorname{Sn}(k, u)}{du} = \surd(1-x^2)\surd(1-k^2x^2) = \operatorname{Cn}(k, u)\operatorname{Dn}(k, u)$$

and the Dn function has now made its appearance. From

$$\operatorname{Cn}(k, u) = \surd\{1 - \operatorname{Sn}^2(k, u)\},$$

$$\frac{d\operatorname{Cn}(k, u)}{du} = \frac{-2\operatorname{Sn}(k, u)\operatorname{Cn}(k, u)\operatorname{Dn}(k, u)}{2\surd\{1-\operatorname{Sn}^2(k, u)\}} = -\operatorname{Sn}(k, u)\operatorname{Dn}(k, u).$$

To get the derivative of $\operatorname{Tn}(k, u)$ consider

$$u = \operatorname{Tn}^{-1}x = \int_0^x \frac{d\mu}{\surd\{(1+\mu^2)(1+k'^2\mu^2)\}}, \qquad x = \operatorname{Tn}(k, u).$$

$$\begin{aligned}
\therefore \frac{dx}{du} = \frac{d\operatorname{Tn}(k, u)}{du} &= \surd\{(1+x^2)(1+(1-k^2)x^2)\} \\
&= \surd[\operatorname{Nc}^2(k, u)\{\operatorname{Nc}^2(k, u) - k^2\operatorname{Tn}^2(k, u)\}] \\
&= \operatorname{Nc}^2(k, u)\sqrt{\left[1 - k^2\frac{\operatorname{Tn}^2(k, u)}{\operatorname{Nc}^2(k, u)}\right]} \\
&= \operatorname{Nc}^2(k, u)\surd\{1 - k^2\operatorname{Sn}^2(k, u)\} \\
&= \operatorname{Nc}^2(k, u)\operatorname{Dn}(k, u).
\end{aligned}$$

Notice that the derivatives of these functions are completely analogous to the derivatives of the trigonometric functions *except* that they are multiplied by $\mathrm{Dn}(k, u)$ in all cases.

One immediately sees another important difference between these functions and the trigonometric functions. Whereas the derivatives of the Sine and Cosine are respectively Cosine and Negative Sine, and hence finding the nth derivative of each is no problem, this fact is not true of the elliptic functions. This, unfortunately, is the price we must pay for the versatility of these functions.

Finally,

$$\frac{d}{du}\mathrm{Dn}(k, u) = \frac{d}{du}\sqrt{(1 - k^2\mathrm{Sn}^2u)}$$

$$= \frac{-2k^2\mathrm{Sn}(k, u)\mathrm{Cn}(k, u)\mathrm{Dn}(k, u)}{2\sqrt{(1 - k^2\mathrm{Sn}^2u)}} = - k^2\mathrm{Sn}(k, u)\mathrm{Cn}(k, u).$$

Let us now investigate a group of nonlinear differential equations satisfied by these functions. These equations arise, for example, in the analysis of electric circuits with saturating elements.

3. Some Differential Equations Involving Elliptic Functions.

3a. *Consider*: $\ddot{x}+ax-bx^3=0$, $x(0)=X_0$, $\dot{x}(0)=0$, $a, b, >0$.

Let $x=X_0\mathrm{Sn}(k, \omega t+K)$. Then

$$\begin{aligned}
\dot{x} &= \omega X_0\mathrm{Cn}\mathrm{Dn}\\
\ddot{x} &= \omega^2X_0[-\mathrm{Sn}\mathrm{Dn}^2 - k^2\mathrm{Sn}\mathrm{Cn}^2]\\
&= - \omega^2X_0[\mathrm{Sn}\mathrm{Dn}^2 + k^2\mathrm{Sn}\mathrm{Cn}^2]\\
&= - \omega^2X_0\mathrm{Sn}[(1 - k^2\mathrm{Sn}^2) + k^2(1 - \mathrm{Sn}^2)]\\
&= - \omega^2X_0\mathrm{Sn}[1 + k^2 - 2k^2\mathrm{Sn}^2]\\
&= - \omega^2X_0(1 + k^2)\mathrm{Sn} + 2k^2\omega^2X_0\mathrm{Sn}^3.
\end{aligned}$$

Substituting in the differential equation, we find that

$$a = \omega^2(1 + k^2) \qquad b = \frac{2k^2\omega^2}{X_0^2}$$

from which it follows that $\omega^2=a-\frac{1}{2}bX_0^2$ and $k^2=bX_0^2/(2a-bX_0^2)$.

Note that for this case we must put a constraint on the variables, namely, $k^2\leqq 1$. Therefore the limiting case is

$$1 = \frac{bX_0^2}{2a - bX_0^2}\cdot$$

Hence $2a-bX_0^2=bX_0^2$ $a=bX_0^2$ and $X_0\leqq\sqrt{(a/b.)}$

If $X_0\geqq\sqrt{(a/b)}$ the motion is no longer periodic. A phase portrait of this equation has a vortex at (0, 0) and saddles at $(\pm\sqrt{(a/b)}, 0)$. Hence for an initial displacement $X_0\geqq\sqrt{(a/b)}$ the system is nonoscillatory.

A further look at the equations for ω^2 and k^2 shows how these parameters depend on the initial condition. If $b=0$, the equation is linear, and $\omega^2=a$, $k^2=0$. As b increases, ω^2 decreases and k^2 increases until $b=a/X_0^2$, at which time oscillations cease. This seems to be one of the peculiar features of nonlinear differential equations, since if a solution to a linear constant coefficient equation is oscillatory for one initial condition, it is oscillatory for all initial conditions.

3b. *Consider*: $\ddot{x}+ax+bx^3=0$, $x(0)=X_0$, $\dot{x}(0)=0$, $a, b>0$.

Let $x=X_0\text{Cn}(k, \omega t)$. Going through the same process as before, we find that

$$k^2 = \frac{bX_0^2}{2(a+bX_0^2)} \quad \text{and} \quad \omega^2 = a + bX_0^2.$$

For this case there is no upper limit on "b" or on "X_0," since

$$\lim_{b\to\infty} k^2 = \frac{1}{2} \quad \text{and} \quad \lim_{X_0^2\to\infty} k^2 = \frac{1}{2}$$

and the solution is oscillatory for all a, b, X_0. Note that the frequency gets greater and greater as "b" and "X_0" are increased.

This conclusion is also verified by a phase portrait. The one singularity is a vortex at (0, 0) and there exists a continuum of closed trajectories about this vortex.

3c. *Consider:* $\ddot{x}-ax+bx^3=0$, $x(0)=X_0$, $\dot{x}(0)=0$, $a, b, >0$.

The solution to this equation is

$$x = X_0\text{Dn}(k, \omega t), \qquad k^2 = 2\left(1 - \frac{a}{bX_0^2}\right), \qquad \omega^2 = \frac{bX_0^2}{2}.$$

Inspection of the potential energy curve shows that there is a saddle at (0, 0) and that there are vortices at $(\pm\sqrt{(a/b)}, 0)$. The phase portrait is shown in Figure 4.

Hence, there is a stable oscillation about $|X_0|=\sqrt{(a/b)}$. If $|X_0|^2=a/b$, $k^2=0$, and there is no motion. This is represented in normalized form in Figure 3c, i.e., $\text{Dn}(0, u)=1$.

If $X_0^2=2a/b$, $k^2=1$, and we are on the separatrix, hence taking infinite time to get to (0, 0).

If $X_0^2>2a/b$, a periodic solution exists, but it is no longer an elliptic function. This periodic solution is represented by the phase trajectory surrounding all the singularities.

Hence, the solution to

$$\ddot{x} - ax + bx^3 = 0, \qquad x(0) = X_0,\ \dot{x}(0) = 0, \qquad a, b > 0$$

is $x=X_0\text{Dn}(k, \omega t)$ where

$$k^2 = 2\left(1 - \frac{a}{bX_0^2}\right) \quad \text{and} \quad \omega^2 = \frac{bX_0^2}{2}$$

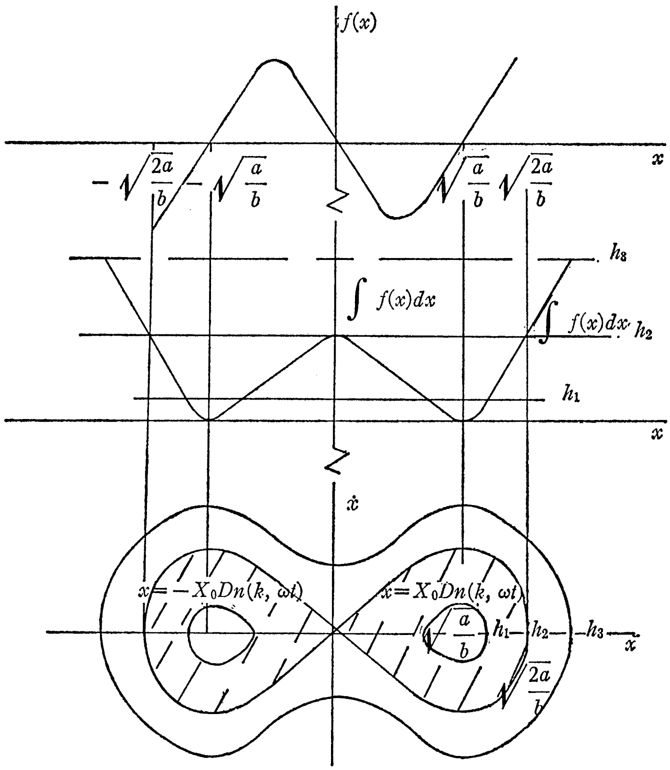

FIG. 4. Phase portrait of $\ddot{x}-ax+bx^3=0$.

for

$$0 < X_0 < \sqrt{\frac{2a}{b}} \quad \text{and} \quad |X_0| \neq \sqrt{\frac{a}{b}}\,\cdot$$

3d. *Finally, consider*: $\ddot{x}-ax-bx^3=0$, $x(0)=X_0$, $\dot{x}(0)=0$, $a,b>0$.

The solution to this equation is

$x = X_0\mathrm{Tn}(k, \omega t+\alpha)$, where $\mathrm{Tn}\,(k,\alpha) = 1$; $\dot{x} = \omega X_0 \mathrm{Nc}^2\mathrm{Dn}$,

$$\ddot{x} = \omega^2 X_0[2\mathrm{Nc}(\mathrm{NcTn})\mathrm{Dn}^2 + \mathrm{Nc}^2(-k^2\mathrm{SnCn})] = \omega^2 X_0 \mathrm{Tn}\left[2\mathrm{Nc}^2\mathrm{Dn}^2 - k^2\frac{\mathrm{NcSn}}{\mathrm{Tn}}\right]$$

$$= \omega^2 X_0\mathrm{Tn}\left[\frac{2(1-k^2\mathrm{Sn}^2)}{\mathrm{Cn}^2} - k^2\right] = \omega^2 X_0 \mathrm{Tn}[2\mathrm{Nc}^2 - 2k^2\mathrm{Tn}^2 - k^2].$$

But $\mathrm{Nc}^2 = 1+\mathrm{Tn}^2$ and therefore

$$\ddot{x} = \omega^2 X_0\mathrm{Tn}[2 + 2(1-k^2)\mathrm{Tn}^2 - k^2] = (2-k^2)\omega^2 X_0\mathrm{Tn} + 2(1-k^2)\omega^2 X_0 \mathrm{Tn}^3.$$

Substituting into the equation, we find that

$$a = (2 - k^2)\omega^2, \qquad b = \frac{2\omega^2(1 - k^2)}{X_0^2}$$

from which we obtain

$$k^2 = 1 - \frac{bX_0^2}{2a - bX_0^2}, \qquad \omega^2 = a - \frac{bX_0^2}{2}.$$

Inspection of the equation shows that there is only one singularity, a saddle at (0, 0), and hence the solution is unstable for all initial conditions. However, for the solution to be $\mathrm{Tn}(k, \omega t)$, we require that $0 \leqq k^2 \leqq 1$. From the equation for k^2, we find that

$$\text{For } |X_0| = \sqrt{\frac{a}{b}},\ k^2=0;\ \text{for } |X_0| = 0,\ k^2=1;\ \text{and for } |X_0| > \sqrt{\frac{a}{b}},\ k^2<0.$$

It can be shown, by a suitable transformation [2], that for the case $k^2<0$, an elliptic tangent of different modulus and frequency arises.

4. Conclusion. We have developed the Jacobian Elliptic Functions from the Elliptic Integral of the First Kind, and have discussed their behaviour as the modulus "k" varies.

We have shown that differential equations of the form

$$\ddot{x} \pm ax \pm bx^3 = 0, \qquad x(0) = X_0, \qquad \dot{x}(0) = 0$$

can be satisfied by the Jacobian elliptic functions. The cases examined have revealed the following results:

(i) For $a>0$, $b<0$, $x=X_0\mathrm{Sn}(k, \omega t+K)$, where $k^2=bX_0^2/(2a-bX_0^2)$ and $\omega^2=a-\frac{1}{2}bX_0^2$. The restriction is that $|X_0| < \sqrt{(a/b.)}$

(ii) For $a>0$, $b>0$, $x=X_0\mathrm{Cn}(k, \omega t)$, where $k^2=bX_0^2/(2(a+bX_0^2))$ and $\omega^2=a+bX_0^2$. There are no restrictions.

(iii) For $a<0$, $b>0$, $x=X_0\mathrm{Dn}(k, \omega t)$, where $k^2=2(1-a/bX_0^2)$ and $\omega^2=bX_0^2/2$. Here, the oscillation is about $x=\pm\sqrt{(a/b)}$, and the restriction is

$$|X_0| < \sqrt{\frac{2a}{b}} \quad \text{and} \quad |X_0| \neq \sqrt{\frac{a}{b}}$$

(iv) For $a<0$, $b<0$, $x=X_0\mathrm{Tn}(k, \omega t)$, where $k^2=1-bX_0^2/(2a-bX_0^2)$ and $\omega^2=a-bX_0^2/2$. The restriction is $|X_0| \leqq \sqrt{(a/b)}$.

By making suitable transformations, it can be shown that the other Jacobian Elliptic Functions satisfy these equations for other ranges of X_0. This is left as an exercise for those curious.

References

1. W. J. Cunningham, An introduction to nonlinear analysis, McGraw-Hill, New York, 1958, p. 80.

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A THEOREM IN APPLIED LINEAR ALGEBRA

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1. Introduction. Let $Ax=u$ be a linear system of m equations

$$(1)\qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1r}x_r &= u_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2r}x_r &= u_2 \\ \cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mr}x_r &= u_m \end{aligned}$$

in $r>m$ unknowns, having a row-regular coefficient matrix:

$$(2)\qquad \text{Rank } A = m = \text{number of equations.}$$

We are concerned with the problem of how to investigate the consistency of an arbitrarily given single linear equation

$$(3)\qquad a_{m+1,1}x_1 + a_{m+1,2}x_2 + \cdots + a_{m+1,r}x_r = u_{m+1}$$

with respect to (1).

The method mostly used in practice is the Gaussian process of successive eliminations. Thus we use the m equations (1) as pivotal equations in order to eliminate m unknowns in (3) to give a reduced equation

$$(4)\qquad a'_{m+1,1}x_1 + a'_{m+1,2}x_2 + \cdots + a'_{m+1,r}x_r = u'_{m+1}$$

with at least m vanishing coefficients. The most interesting case is when not only m but all coefficients vanish in (4) and we obtain a reduced equation

$$(5)\qquad 0\cdot x_1 + 0\cdot x_2 + \cdots + 0\cdot x_r = u'_{m+1}.$$

In this paper we answer the question of what is the numerical value of the right hand side (u'_{m+1}) of an equation (5) with vanishing coefficients. Thus we shall show that u'_{m+1} is the difference

$$(6)\qquad u'_{m+1} = u_{m+1} - (a_{m+1,1}x_{01} + a_{m+1,2}x_{02} + \cdots + a_{m+1,r}x_{0r})$$

where the vector $(x_{01}, x_{02}, \cdots, x_{0r})$ is that solution of (1) which corresponds to the principle of least squares

$$(7)\qquad \sum x_i^2 = \text{a minimum.}$$

2. The right hand side of an orthogonalized equation. We consider the equation (5) as an equation (4) whose coefficient vector $(a'_{m+1,1}, a'_{m+1,2}, \cdots, a'_{m+1,r})$ is orthogonal to the coefficient vector $(a_{i1}, a_{i2}, \cdots, a_{ir})$ of every given equation in (1):

$$(8)\qquad a_{i1}a'_{m+1,1} + a_{i2}a'_{m+1,2} + \cdots + a_{ir}a'_{m+1,r} = 0$$

for $i=1, 2, \cdots, m$, or

$$(9) \qquad A\begin{bmatrix} a'_{m+1,1} \\ a'_{m+1,} \\ \cdot \\ a'_{m+1,r} \end{bmatrix} = 0.$$

Let $q_1, q_2, \cdots, q_m$ be m arbitrary multipliers. We multiply both sides of each given equation in (1) by the corresponding multiplier and add to the additional equation (3) which reduces to (4). We have

$$(10) \qquad A^T\begin{bmatrix} q_1 \\ q_2 \\ \cdot \\ q_m \end{bmatrix} + \begin{bmatrix} a_{m+1,1} \\ a_{m+1,2} \\ \cdot \\ a_{m+1,m} \\ \cdot \\ a_{m+1,r} \end{bmatrix} = \begin{bmatrix} a'_{m+1,1} \\ a'_{m+1,2} \\ \cdot \\ a'_{m+1,m} \\ \cdot \\ a'_{m+1,r} \end{bmatrix}$$

in which A^T is the transpose of A. The right hand side, u'_{m+1}, of (4) is

$$(11) \qquad u'_{m+1} = u_{m+1} + \sum_1^m q_i u_i.$$

Now let us pre-multiply both sides of (10) by the coefficient matrix A of (1):

$$(12) \qquad AA^T\begin{bmatrix} q_1 \\ q_2 \\ \cdot \\ q_m \end{bmatrix} + A\begin{bmatrix} a_{m+1,1} \\ a_{m+1,2} \\ \cdot \\ a_{m+1,m} \\ \cdot \\ a_{m+1,r} \end{bmatrix} = A\begin{bmatrix} a'_{m+1,1} \\ a'_{m+1,2} \\ \cdot \\ a'_{m+1,m} \\ \cdot \\ a'_{m+1,r} \end{bmatrix}.$$

Now AA^T is a square symmetric matrix and, according to (2), it is also a nonsingular (actually a positive definite) matrix. Now we require such multipliers q_i that the new equation (4) satisfies the orthogonality condition (9). Therefore

$$(13) \qquad AA^T\begin{bmatrix} q_1 \\ q_2 \\ \cdot \\ q_m \end{bmatrix} = -A\begin{bmatrix} a_{m+1,1} \\ a_{m+1,2} \\ \cdot \\ a_{m+1,m} \\ \cdot \\ a_{m+1,r} \end{bmatrix},$$

or

(14) $$[q_1\ q_2 \cdots q_m]AA^T = -[a_{m+1,1}\ a_{m+1,2} \cdots a_{m+1,r}]A^T.$$

Let us apply the principle of least squares (7) to the consistent linear system (1). According to the well-known routine ("method of correlatives") we first define m numbers $k_1, k_2, \cdots, k_m$, called "correlatives," as follows:

(15) $$AA^T\begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ k_m \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ u_m \end{bmatrix}.$$

The $k_1, k_2, \cdots, k_m$ being found, we calculate the required "minimum" solution $(x_{01}, x_{02}, \cdots, x_{0r})$ by means of the formula

(16) $$\begin{bmatrix} x_{01} \\ x_{02} \\ \cdot \\ x_{0m} \\ \cdot \\ x_{0r} \end{bmatrix} = A^T\begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ k_m \end{bmatrix}.$$

Let us post-multiply both sides of (14) by the column-vector k in the m correlatives $k_1, k_2, \cdots, k_m$. Taking (15) and (16) into consideration we obtain

(17) $$[q_1\ q_2 \cdots q_m]\begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ u_m \end{bmatrix} = -[a_{m+1,1}\ a_{m+1,2} \cdots a_{m+1,r}]\begin{bmatrix} x_{01} \\ x_{02} \\ \cdot \\ x_{0m} \\ \cdot \\ x_{0r} \end{bmatrix}$$

or

(18) $$\sum_1^m q_i u_i = -(a_{m+1,1}x_{01} + a_{m+1,2}x_{02} + \cdots + a_{m+1,r}x_{0r}).$$

Substituting into (11) we obtain formula (6). In this way we have proved the following:

THEOREM. *Let us orthogonalize the additional equation* $a_{m+1,1}x_1 + a_{m+1,2}x_2 + \cdots + a_{m+1,r}x_r = u_{m+1}$, $r > m$, *with respect to the preceding* m *equations*

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ir}x_r = u_i, \qquad i = 1, 2, \cdots, m$$

with a row-regular coefficient matrix. The right hand side u'_{m+1} of the orthogonalized equation

$$a'_{m+1,1}x_1 + a'_{m+1,2}x_2 + \cdots + a'_{m+1,r}x_r = u'_{m+1}$$

is exactly the difference

$$u'_{m+1} = u_{m+1} - (a_{m+1,1}x_{01} + a_{m+1,2}x_{02} + \cdots + a_{m+1,r}x_{0r}),$$

where the vector $(x_{01}, x_{02}, \cdots, x_{0r})$ is that common solution of the preceding m equations which corresponds to the principle of least squares.

Now assume that the coefficient vector of (3) is linearly dependent on the coefficient vectors of (1). In this case there are m multipliers $q_1, q_2, \cdots, q_m$ satisfying the r linear equations

$$A^T\begin{bmatrix} q_1 \\ q_2 \\ \cdot \\ q_m \end{bmatrix} + \begin{bmatrix} a_{m+1,1} \\ a_{m+1,2} \\ \cdot \\ a_{m+1,m} \\ \cdot \\ a_{m+1,r} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ \cdot \\ 0 \end{bmatrix} \tag{19}$$

and these m "elimination multipliers" $q_1, q_2, \cdots, q_m$ are exactly the same as the m "orthogonalization multipliers" $q_1, q_2, \cdots, q_m$ in (13). This means that if we reduce (Gaussian elimination) or if we orthogonalize (for example by the Schmidt process of successive orthogonalizations), we obtain one and the same new equation (5) with vanishing coefficients to which our theorem can be applied immediately.

3. A new interpretation of a well-known theorem. Let $r=m$ in (1). Condition (2) would mean that the square matrix is nonsingular and the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= u_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= u_2 \\ \cdots\cdots\cdots\cdots & \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mm}x_m &= u_m \end{aligned} \tag{20}$$

has a unique solution $(x_{01}, x_{02}, \cdots, x_{0m})$. Using the above m equations as pivotal equations we eliminate all m unknowns in any arbitrarily given additional equation

$$a_{m+1,1}x_1 + a_{m+1,2}x_2 + \cdots + a_{m+1,m}x_m = u_{m+1} \tag{21}$$

which necessarily reduces to an equation

$$(22) \qquad 0\cdot x_1 + 0\cdot x_2 + \cdots + 0\cdot x_m = u'_{m+1}$$

with vanishing coefficients. There is a well-known theorem [1] that

$$(23) \qquad u'_{m+1} = u_{m+1} - (a_{m+1,1}x_{01} + a_{m+1,2}x_{02} + \cdots + a_{m+1,m}x_{0m}).$$

It is easy to see that the method of correlatives, as described by formulae (15) and (16), can be applied also to a linear system (20) with a square regular coefficient matrix, in which case the vector x_0 is the unique solution of the given linear system $Ax=u$. It follows that all formulae in section 2 are also valid if $m=r$. Therefore, the theorem in the previous section can be generalized to be valid also in the case $m=r$, but we may say that the vector x_0 is that solution of the given linear system $Ax=u$ which has been evaluated by the method of correlatives. Orthogonalizing (21) we necessarily obtain an equation (22) with vanishing coefficients, this allowing another interpretation of the above theorem.

4. Numerical examples. a) Find the shortest distance between the two parallel planes

$$1. \quad x + 2y - z = 6,$$
$$2. \quad 3x + 6y - 3z = 20.$$

We multiply the first equation by $q_1=-3$ and add to the second equation which becomes

$$2'. \quad 0 = 2.$$

According to our theorem, $2=20-(3x_0+6y_0-3z_0)$ in which $P_0(x_0, y_0, z_0)$ is the point on (1) with the shortest distance, $OP^2=x^2+y^2+z^2=$ a minimum, from the origin. Therefore the required distance is $d=2/\sqrt{54}$, $(54=3^2+6^2+3^2)$.

b) Let $P_0(x_0, y_0, z_0)$ be the point of intersection of the plane

$$1. \quad x + 2y - z = 6$$

with the perpendicular from the origin. Find the shortest distance of P_0 from the plane

$$2. \quad x + 5y - z = 13.$$

We multiply the first equation by $q_1=-2$ and add to the second equation which turns to an orthogonalized equation

$$2'. \quad -x + y + z = 1$$
$$(+1)(-1) + (+2)(+1) + (-1)(+1) = 0.$$

The answer is $d=1/\sqrt{27}$.

Reference

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A DIVISION ALGORITHM

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Here is a new algorithm which tests for divisibility. Whenever divisibility of a number N by a number P occurs, this algorithm finds the quotient without estimating trial divisors (or circumventing this difficulty by repeated subtraction of the divisor from the dividend).

First, one may make the following observation: the cases where

$$P \equiv 0 \pmod 2 \quad \text{and/or} \quad P \equiv 0 \pmod 5 \tag{1}$$

are immediately detected by inspection of the last digit, and—if we wish to stipulate that $N \equiv 0 \pmod P$, N would likewise have to have property (1). It is therefore without loss of generality that cases (1) be excluded since this condition can always be fulfilled by dividing both N and P by 2 and/or 5 (or by powers of these numbers)—if necessary.

Let u_P be the unit digit of P, and write $P = 10d + u_P$. Then $u_P \in \{1, 3, 7, 9\}$. We now wish to obtain a multiple of P, say kP, that is to have the form $kP = 10n \pm 1$, where n is a natural number. We will refer to the $(10n+1)$-form as case I, and will speak of case II whenever the form $10n-1$ appears.

The following chart discusses the different possibilities.

The unit-digit of the divisor P	The required multiplier	Case	The number n which occurs in the form $10n \pm 1$	The difference between the divisor P and the number n
u_P	k		n	$P-n$
1	1	I	d	$9d+1$
3	3	II	$3d+1$	$7d+2$
7	3	I	$3d+2$	$7d+5$
9	1	II	$d+1$	$9d+8$

The fourth column (values of n) is obtained by performing the indicated operations. Thus, for instance, line 3 would be found by realizing that $P = 10d + 7$. Therefore $kP = 3P = 30d + 21 = 10(3d+2)+1$, and thus, case I $(10n+1)$ is at hand with n equaling $3d+2$.

One may also note that under all circumstances $n < P$, considerably so if d is a "fairly large" number (see column 5).

Since $N \equiv 0 \pmod P$, $N = PQ$, where Q is the desired quotient. It follows that $kN = Q(kP)$, or, $kN = Q(10n \pm 1)$.

It is convenient to write Q in slightly different form in the two cases. For case I, let $Q = \sum_{i=0}^{r} 10^i c_i$, where $0 \leqq c_i \leqq 9$; for case II, $Q = 10^{r+1} - \sum_{i=0}^{r} 10^i c_i$ with the same restrictions on c_i. Then, in case I,

$$kN = Q(10n+1) = \sum_{i=0}^{r} 10^i c_i(10n+1)$$

$$= c_0 + 10\left[\sum_{i=0}^{r-1} 10^i(c_{i+1}+nc_i) + 10^r nc_r\right].$$

Now we start an iterative process as seen below, whereby the numbers obtained in the successive steps will be called st_k for $k=1, 2, 3, \cdots, (r+1)$, where $r+1$ equals the number of digits in the quotient. This process is described as follows:

$$\text{st}_1 = \frac{kN - c_0}{10} - nc_0 = c_1 + 10\left[\sum_{i=1}^{r-1} 10^{i-1}(c_{i+1}+nc_i) + 10^{r-1}nc_r\right]$$

$$\text{st}_2 = \frac{\text{st}_1 - c_1}{10} - nc_1 = c_2 + 10\left[\sum_{i=2}^{r-1} 10^{i-2}(c_{i+1}+nc_i) + 10^{r-2}nc_r\right]$$

$$\cdots \qquad \cdots$$

$$\text{st}_k = \frac{\text{st}_{k-1} - c_{k-1}}{10} - nc_{k-1} = c_k + 10\left[\sum_{i=k}^{r-1} 10^{i-k}(c_{i+1}+nc_i) + 10^{r-k}nc_r\right] \tag{2}$$

$$\cdots \qquad \cdots$$

$$\text{st}_{r-1} = \frac{\text{st}_{r-2} - c_{r-2}}{10} - nc_{r-2} = c_{r-1} + 10\left[\sum_{i=r-1}^{r-1} 10^{i-r+1}(c_{i+1}+nc_i) + 10nc_r\right]$$

$$= c_{r-1} + 10[(c_r + nc_{r-1}) + 10nc_r]$$

$$\text{st}_r = \frac{\text{st}_{r-1} - c_{r-1}}{10} - nc_{r-1} = c_r + 10nc_r$$

$$\text{st}_{r+1} = \frac{\text{st}_r - c_r}{10} - nc_r = 0.$$

When a similar reasoning chain is followed in case II, we obtain

$$\text{st}_k = \frac{\text{st}_{k-1} - c_{k-1}}{10} + nc_{k-1}$$

$$= c_k + 10\left[\sum_{i=k}^{r-1} 10^{i-k}(c_{i+1}-nc_i) + 10^{r-k}(-nc_r + 10n - 1)\right]$$

for $k=1, 2, 3, \cdots, (r+1)$. Then,

$$\text{st}_{r+1} = \frac{\text{st}_r - c_r}{10} + nc_r = 10n - 1 = kP.$$

Thus, in case I, the value of st_{r+1}, the number in the $(r+1)$-st step, is zero, whereas case II exhibits the number $10n-1$ which is equal to kP. Should one overlook the occurrence of this number, a repetitive phenomenon will direct one's attention to it. This is seen from forming st_{r+2}, and observing that

$$\mathrm{st}_{r+2} = \frac{\mathrm{st}_{r+1} - c_{r+1}}{10} + nc_{r+1} = \frac{10n - 1 - 9}{10} + 9n$$

since the last digit (c_{r+1}) of a number of the form $10n-1$ must be 9. This value, however, simplifies to $10n-1$, and, thus, $\mathrm{st}_{r+m}=10n-1$ for $m=2, 3, 4, \cdots$.

We are now ready to formulate the divisibility test. To investigate if $N\equiv 0 \pmod P$, $P\not\equiv 0 \pmod 2$ and $P\not\equiv 0 \pmod 5$ may be assumed without loss of generality. Then kP will always be of the form $10n\pm 1$, with k being 1 or 3. The above chart explains this statement in detail to say that $k=1$, whenever P ends in 1 or 9, and $k=3$, if the last digit of P is 3 or 7. The form $10n+1$ (case I) occurs for the end-digits 1 and 7. For the other two possibilities (end-digits 3 or 9), the form $10n-1$ (case II) will be reached.

Then, in case I, the product of n and the last digit of kN must be subtracted from the number consisting of all but the last digit of kN. This process must be repeated as many times as the number of digits in the quotient. At that stage, if and only if $N\equiv 0 \pmod P$, zero is reached.

In case II, a similar procedure is followed, except that subtraction is to be replaced by addition. Here the process is again repeated as many times as the number of digits in the quotient. At that stage, if and only if $N\equiv 0 \pmod P$, the number $kP=10n-1$ is reached. If overlooked, this number recurs indefinitely.

In either case, if the desired outcome is not obtained, nondivisibility must be concluded. In case of divisibility the quotient Q may also be "read off." In case I, it is a number consisting of the "removed" digits to be read from the final stage to the initial. This is seen from the fact that the "removed" digits are the c_i ($i=0, 1, 2, \cdots, r$) in this order, and Q was stipulated as being the number $Q=\sum_{i=0}^{r} 10^i c_i$ (see relationship (2)).

In case II—owing to the different form of Q—the quotient is found by subtracting the number obtained by reading the "removed" digits (in the same manner as in case I) from 10^{r+1}, where $r+1$ represents the number of steps taken (which equals the number of digits in the quotient).

If one compares this divisional algorithm with "long division," the same number of steps is needed, and each step entails multiplication and subtraction (or addition). However, the proposed method differs from ordinary "long division" in two respects: (1) no estimate of trial divisors is necessary, and (2) the multiplication (obtaining of partial products) will concern smaller numbers. In the usual method, these partial products will be Pc_i ($i=0, 1, 2, \cdots, r$). In the above device, they are nc_i ($i=0, 1, 2, \cdots, r$). As seen from the chart, $n<P$, and the larger the value of d, the more will this simplification make itself felt.

Three examples will illustrate the method.

Example 1. The number 879,336 is to be tested for divisibility by 1,357, and —in case of divisibility—the quotient is to be found. Here, $d=135$, $u_P=7$. Therefore, $k=3$, $n=3d+2=407$ and case I is at hand.

Proposed method	"Long division"
$kN = 263800 \mid 8$	648
3256	$1357\overline{)879336}$
$26054 \mid 4$	8142
1628	6513
$2442 \mid 6$	5428
2442	10856
----	10856

Thus, $N \equiv 0 \pmod{P}$ and $Q = 648$.

Example 2. The same problem as above, except that the numbers 245,745 and 645 are involved. Here, since the greatest common divisor of 245,745 and 645 is 5, $\overline{N} = 49{,}149$ and $\overline{P} = 129$, and, therefore, $d = 12$, $u_P = 9$. Hence, $k = 1$, $n = d + 1 = 13$, and case II occurs.

Proposed method	"Long division"
$k\overline{N} = 4\,9\,1\,4 \mid 9$	381
$1\,1\,7$	$645\overline{)245745}$
$5\,0\,3 \mid 1$	1935
$1\,3$	5224
$5\,1 \mid 6$	5160
$7\,8$	645
$1\,2\,9 = \overline{P} = 10n - 1$ (if continued: 129, 117, 129, . . .)	645

Therefore, divisibility is concluded, and $Q = 10^3 - 619 = 381$.

Example 3. If $N = 18{,}024$ and $P = 783$, then $d = 78$, $u_P = 3$, hence, $k = 3$, $n = 3d + 1 = 235$, and case II is at hand. Here,

$$
\begin{array}{rl}
kN = 5\,4\,0\,7 & \mid 2 \\
4\,7\,0 & \\
\hline
5\,8\,7 & \mid 7 \\
1\,6\,4\,5 & \\
\hline
2\,2\,3\,2 & = st_{r+1} \neq 10n - 1.
\end{array}
$$

Hence, $N \not\equiv 0 \pmod{P}$ must be concluded.

As illustrated in example 3, whenever the desired number (zero in case I, $10n-1$ in case II) does not appear in the $(r+1)$-st step, the division in question leaves a nonvanishing remainder R, and this remainder cannot be "read off" from the above device, nor—in this case—the quotient. However, we conjecture that

$$\text{st}_{r+1} \equiv (\text{st}_{r+1}{}' - \text{st}_{r+1})R \pmod{P} \tag{3}$$

where $\text{st}_{r+1}{}'$ is the number obtained by setting up the divisional device for $N' = N+1$. If this is true, congruence (3) will determine the value of the remainder R, and our algorithm, applied to the number $N-R$ will lead to the desired quotient.

Example 4. In example 3, $N' = 18{,}025$. Therefore,

$$\begin{array}{rr|l} kN' = & 5\,4\,0\,7 & 5 \\ & \underline{1\,1\,7\,5} & \\ & 6\,5\,8 & 2 \\ & \underline{4\,7\,0} & \\ & 1\,1\,2\,8 & = \text{st}_{r+1}{}'. \end{array}$$

Hence, $2232 \equiv -1104R \pmod{783}$, or $462R \equiv 666 \pmod{783}$. This congruence has three solutions, namely, $R_1 = 15$, $R_2 = 276$, and $R_3 = 537$. However, since $N - R_i \not\equiv 0 \pmod{P}$ for $i = 2$ or 3, R_2 and R_3 must be discarded. (Obviously, whenever congruence (3) has only one solution, this particular difficulty does not arise.) Therefore, $R = 15$ and $\overline{N} = N - R = 18{,}009$. Then,

$$\begin{array}{rr|l} k\overline{N} = & 5\,4\,0\,2 & 7 \\ & \underline{1\,6\,4\,5} & \\ & 7\,0\,4 & 7 \\ & \underline{1\,6\,4\,5} & \\ & 2\,3\,4\,9. & \end{array}$$

Therefore, $Q = 100 - 77 = 23$.

THE DISTANCE OF A POINT FROM A LINE

T. A. BROWN, The RAND Corporation, Santa Monica, California

As Mott [2] points out, mathematicians often have trouble giving a cogent treatment of the sign in the formula for the distance of a point from a line in analytic geometry. Furthermore, derivations of the distance formula itself sometimes seem unduly cumbersome and entail discussion of several cases. This paper will present a simple treatment inspired by Klein [1] which is in some respects more elegant than the treatments usually seen.

As illustrated in example 3, whenever the desired number (zero in case I, $10n-1$ in case II) does not appear in the $(r+1)$-st step, the division in question leaves a nonvanishing remainder R, and this remainder cannot be "read off" from the above device, nor—in this case—the quotient. However, we conjecture that

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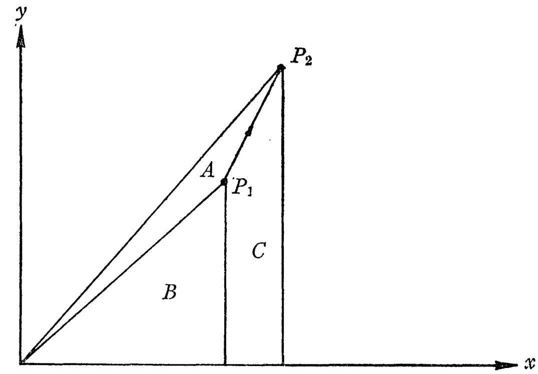

$$A + B + C = \frac{x_2 y_2}{2}$$

$$B = \frac{x_1 y_1}{2}$$

$$C = (x_2 - x_1)\frac{(y_2 + y_1)}{2} = \frac{x_2 y_2}{2} - \frac{x_1 y_1}{2} + \frac{x_2 y_1 - x_1 y_2}{2}$$

$$\therefore\ A = \frac{x_1 y_2 - x_2 y_1}{2}.$$

FIG. 1

Let O denote the origin. Given points P_1: (x_1, y_1) and P_2: (x_2, y_2), define

$$(P_1, P_2) = \frac{x_1 y_2 - x_2 y_1}{2}.$$

A simple computation (see Fig. 1) shows that (P_1, P_2) is the area of the triangle OP_1P_2, with a positive sign if OP_1P_2 are in counter-clockwise order about the triangle and a negative sign if OP_1P_2 are in clockwise order about the triangle. By translating the axes (or by considering Fig. 2), we see that

$$\text{Area } P_1P_2P_3 = (P_1, P_2) + (P_2, P_3) + (P_3, P_1).$$

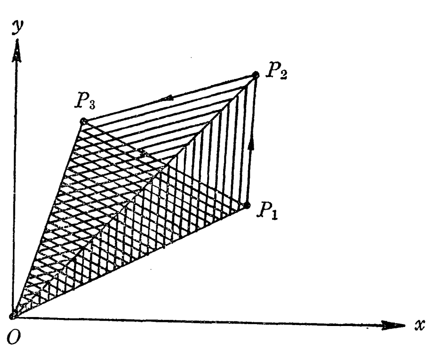

FIG. 2. $OP_1P_2 + OP_2P_3 - OP_3P_1 = P_1P_2P_3$.

In fact, given any polygon, convex or not, with vertices $P_1, P_2, \cdots, P_n, P_1$ (in that order) we see that

$$\text{Area } P_1P_2 \cdots P_n = (P_1, P_2) + (P_2, P_3) + \cdots + (P_{n-1}, P_n) + (P_n, P_1),$$

where the sign is positive if the order of the vertices is counter-clockwise and negative if the order of the vertices is clockwise.

The straight line determined by P_1 and P_2 is the locus of all points P such that the triangle P_1P_2P has zero area. Hence the equation of the line P_1P_2 is

$$(P_1, P_2) + (P_2, P) + (P, P_1) = 0.$$

Reversing the order of P_1 and P_2 changes the sign of all the terms in the equation. Denote the distance from P_1 to P_2 by $d(P_1, P_2) = +\sqrt{\{(x_1-x_2)^2 + (y_1-y_2)^2\}}$. Then if P_3 is any point, distance from P_3 to line P_1P_2 is

$$\frac{2 \times \text{Area } P_1P_2P_3}{d(P_1, P_2)} = \frac{(P_1, P_2) + (P_2, P_3) + (P_3, P_1)}{\frac{1}{2}d(P_1, P_2)}.$$

If the line P_1P_2 is thought of as having orientation from P_1 to P_2, then this formula clearly gives a positive value if P_3 is to the left of the line, and a negative value if P_3 is to the right of the line.

Any views expressed in this paper are those of the author. They should not be interpreted as reflecting the views of the RAND Corporation or the official opinion or policy of any of its governmental or private research sponsors.

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ANCIENT ALGORITHMS ADAPTED TO MODERN COMPUTERS

LADIS D. KOVACH, Pepperdine College

Introduction. When we look at the variety and complexity of tasks accomplished by the modern digital computer there is a tendency to be awed. The numerical analyst, the computer designer, and the programmer can feel justifiably proud of the progress they have made in the last two decades. They have developed a number of clever algorithms and logic circuits to enable the computer to solve problems of the greatest difficulty by means of the most simple mathematics.

Faced with this great modern advance we can certainly be excused for getting carried away by our enthusiasm for the contributions made by today's scientists. In the midst of this situation it is easy to overlook the debt we owe to certain ancient cultures. The fact is, however, that a close examination of

In fact, given any polygon, convex or not, with vertices $P_1, P_2, \cdots, P_n, P_1$ (in that order) we see that

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Faced with this great modern advance we can certainly be excused for getting carried away by our enthusiasm for the contributions made by today's scientists. In the midst of this situation it is easy to overlook the debt we owe to certain ancient cultures. The fact is, however, that a close examination of the relationships between the old and the new can also produce some useful results.

It is the purpose of this paper to point out the similarities that exist between some ancient algorithms and some modern techniques. Aside from pointing out the interesting fact that we have "gone full circle," this paper may (hopefully) also suggest some new fields for investigation.

Numeration Systems. Some mathematical techniques used in the ancient world were necessary because of a limited numeration system in use. For example, the Egyptians had no notation for a fraction with a numerator other than unity, with the exception of 2/3. The use of these unit fractions imposed certain restrictions on their arithmetic which, in a sense, is analogous to the restrictions imposed by the hardware in a computer.

The Rhind papyrus, dating from about 1700 B.C., gives a clue regarding the computational methods used by the Egyptians. Preceding the problems on the papyrus there is a table showing how various fractions can be expressed as the sum of unit fractions. Here we find entries like the following:

$$2/7 = 1/4 + 1/28, \quad 2/97 = 1/56 + 1/679 + 1/776, \quad \text{and} \quad 2/99 = 1/66 + 1/198.$$

It is not known what methods were used to derive these decompositions. Since more than one solution is possible, there were undoubtedly many different procedures in use—some of them probably closely-guarded secrets. Thus,

$$\begin{aligned}2/43 &= 1/24+1/258+1/1032 = 1/30+1/86+1/645 = 1/36+1/86+1/172+1/774\\ &= 1/42+1/86+1/129+1/301 = 1/43+1/43\end{aligned}$$

and others. Of these, the Egyptians seem to have preferred

$$2/43 = 1/42 + 1/86 + 1/129 + 1/301.$$

If the fraction had the form a/bc with $b+c=ka$, then the decomposition

$$\frac{a}{bc} = \frac{1}{b\cdot(b+c)/a} + \frac{1}{c\cdot(b+c)/a}$$

was sometimes used.

Let us examine more closely some of the reasons why the Egyptians "seem to have preferred" the decomposition $1/42+1/86+1/129+1/301$ for the fraction $2/43$. We have

$$\begin{aligned}1/42 + 1/86 + 1/129 + 1/301 &= 1/42 + 1/43(1/2 + 1/3 + 1/7)\\ 2/43 &= 1/2\cdot3\cdot7 + 1/43(1/2 + 1/3 + 1/7).\end{aligned}$$

The presence of the integers 2, 3, and 7 at two different places in the above decomposition is more than an accident.

Noting that 43 is a prime number, we are led to the question: Under what conditions can $2/n$ be written as

$$\frac{1}{2\cdot3\cdot5\cdot7\cdot11\cdots p_k} + \frac{1}{n}(1/2 + 1/3 + 1/5 + 1/7 + \cdots + 1/p_k)$$

where n is a given number and p_k is the kth prime (assuming $p_1=2$, $p_2=3$, etc.)? We can solve the equation

$$\frac{2}{n} = \frac{1}{2\cdot 3 \cdots p_k} + \frac{1}{n}(1/2 + 1/3 + \cdots + 1/p_k)$$

for n to obtain the following:

$$n = 2\cdot 2\cdot 3\cdot 5 \cdots p_k - \sum_{i=1}^{k} \prod\nolimits_i 2\cdot 3\cdot 5 \cdots p_k, \qquad (k > 1)$$

where $\prod_i$ means the product $p_1 \cdots p_{i-1}p_{i+1} \cdots p_k$, that is, the ith prime is omitted.

This can be clarified further if we assign values 1, 2, 3, $\cdots$ to k and compute the resulting n. The work can be shown in a nested form as follows:

$$n = 3 = 2\cdot 2 - 1 \qquad \text{for } k = 1$$
$$n = 7 = 3(2\cdot 2 - 1) - 2 \qquad \text{for } k = 2$$
$$n = 29 = 5[3(2\cdot 2 - 1) - 2] - 2\cdot 3 \qquad \text{for } k = 3$$
$$n = 173 = 7\{5[3(2\cdot 2 - 1) - 2] - 2\cdot 3\} - 2\cdot 3\cdot 5 \qquad \text{for } k = 4$$
$$n = 1693 = 11|7\{5[3(2\cdot 2 - 1) - 2] - 2\cdot 3\} - 2\cdot 3\cdot 5| - 2\cdot 3\cdot 5\cdot 7 \qquad \text{for } k = 5$$

Of course, this process does not produce the prime 43 but, assuming that the above scheme was known to the Egyptians, we can ask the next obvious question. Do we need to use all the primes or can some be omitted?

If we omit $p_1=2$, we have $n=5(2\cdot 3-1)-3=22$ which is not prime. If we omit $p_2=3$, we have $n=7[5(2\cdot 2-1)-2]-2\cdot 5=81$ which is not prime. But if we omit $p_3=5$, we have

$$n = 3 = 2\cdot 2 - 1 \qquad \text{for } k = 1$$
$$n = 7 = 3(2\cdot 2 - 1) - 2 \qquad \text{for } k = 2$$
$$n = 43 = 7[3(2\cdot 2 - 1) - 2] - 2\cdot 3 \qquad \text{for } k = 3$$
$$n = 431 = 11\{7[3(2\cdot 2 - 1) - 2] - 2\cdot 3\} - 2\cdot 3\cdot 7 \qquad \text{for } k = 4.$$

Thus, by omitting the third prime, we have obtained 43 in the third step.

The decomposition of 2/97 used by the Egyptians can be written,

$$2/97 = 1/56 + 1/97(1/7 + 1/8) = 1/7\cdot 8 + 1/97(1/7 + 1/8)$$

which seems to use a method similar to the previous ones.

One can conjecture whether a more thorough study of decomposition into unit fractions may lead to a formula for generating prime numbers. This last has been a dream of mathematicians—both amateur and professional—for some time.

Iteration. Ancient mathematicians developed another process which they used to solve a number of difficult problems. This was the process of iteration.

Iteration came about as a result of two conditions that existed in the ancient world. First, because of the absence of sophisticated techniques, it was necessary to use the simple arithmetic that is characteristic of many iterative processes. Second, since human labor and time were not valued as highly as today, mathematicians did not mind expending a great deal of effort in order to arrive at a result. Iteration is a way of trading simplicity and hard work for sophistication, which is also a characteristic of the electronic computer.

As an example of the last idea, consider the problem of finding the square root of 29,929. This can be done by a method developed by Hero of Alexandria over two thousand years ago.

If we "guess" the number 173 and divide 29,929 by 173, we obtain 173, which clearly establishes 173 as the correct solution. Lacking occult powers, however, we would be much more likely to guess 170. Then the division,

$$29{,}929 \div 170 = 176$$

indicates that 170 is too small but 176 is too large. It is quite natural to take for the next trial the arithmetic average of 170 and 176. Another division then establishes the accuracy of this average.

The most important advantages of this method are that its successive steps are all similar and that any desired accuracy can be obtained. For example to find $\sqrt{56}$ with a first guess of 7, the work proceeds as follows:

$$56 \div 7 = 8 \qquad \frac{8+7}{2} = 7.5$$

$$56 \div 7.5 = 7.47 \qquad \frac{7.47+7.5}{2} = 7.485$$

$$56 \div 7.485 = 7.4816 \qquad \frac{7.485+7.4816}{2} = 7.4833$$

$$56 \div 7.4833 \text{ etc.} \qquad \text{—}$$

By carrying out the first four or five steps of the above iteration the reader will discover how much work can result from simple arithmetic operations.

Today, however, the electronic computer is ideally suited to iteration. The "low I.Q." of the computer is compatible with the idea of having many repetitions of the same calculation. The computer's high speed, on the other hand, reduces the drudgery to routine. Thus, iterative methods are undergoing a revival and it might be profitable to look more carefully at some of the schemes used in the ancient world.

As a further example of the power and versatility of the iterative method, consider the problem of finding the reciprocal of a number a without using division. In other words, we wish to find an expression x_n which is nearly $1/a$.

We can make the difference between these two, $1/a-x_n$, small by making $1-ax_n$ small. Assuming that this difference is small we seek an expression x_{n+1} so that $1-ax_{n+1}$ will be smaller still. This can be accomplished by considering

$$1 - ax_{n+1} = k(1 - ax_n)^2 = k - 2akx_n + a^2k\,x_n^2.$$

We can solve this last equation for x_{n+1} easily provided that $k=1$. Then $-ax_{n+1}=-2ax_n+a^2x_n^2$ and $x_{n+1}=x_n(2-ax_n)$. Starting with an initial guess of 0.1, this method produces 0.142848 for the reciprocal of 7 in three steps. The correct value to six places is 0.142857.

In another example of an iterative process, we use an idea based on Newton's method. Newton found that a root of the equation $f(x)=0$, for continuous f, is approximated by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Suppose we wish to obtain an approximate value of the reciprocal square root of a number N, i.e., $1/\sqrt{N}$. Consider the equation

$$f(x) = N - 1/x^2 = 0$$

whose root x is the desired quantity. Since $f'(x)=2/x^3$, we have

$$x_{n+1} = \frac{x_n}{2}(3 - Nx_n^2).$$

We can rewrite this as

$$x_{n+1} = 0.5x_n(3 - Nx_n^2)$$

and obtain a formula for the reciprocal square root which is ideally adapted to computers because it requires no divisions.

Even more remarkable is the fact that one more multiplication can convert this formula to one for finding square root without divisions. Having found $x=1/\sqrt{N}$, multiplication of both members by N produces $\sqrt{N}=Nx$.

As an example of the use of this formula, suppose we wish to find $\sqrt{2}$. We first find $1/\sqrt{2}$ with $N=2$ and, say, $x_0=0.5$. Then we obtain successively,

$$x_1 = 0.5(0.5)[3 - 2(0.5)^2] = 0.625$$
$$x_2 = 0.5(0.625)[3 - 2(0.625)^2] = 0.69336$$
$$x_3 = 0.5(0.69336)[3 - 2(0.69336)^2] = 0.7067085.$$

Multiplying this last equality by 2 gives 1.41342 compared to the actual 1.41421.

More sophisticated iterative schemes can be developed from the simpler ones. As a final example, consider an iteration process developed from Newton's formula,

$$x_{n+1} = \frac{x_n(x_n^2 + 3N)}{3x_n^2 + N}$$

which is a method for computing $\sqrt{N}$. The very rapid rate of convergence of this formula can be demonstrated by an example. To find $\sqrt{10}$ with an initial guess of 3, we have,

$$x_1 = \frac{3(9+30)}{3(9)+10} = 3.16216$$

$$x_2 = \frac{3.16216[(3.16216)^2+30]}{3(3.16216)^2+10} = 3.16227\ 76601\ 68341$$

which is in error by 4 units in the last figure.

Simplified Mathematics. We find a number of simple techniques in use in ancient mathematics. In spite of the gap between these and modern sophistication, we may profit by examining some of the simple methods more closely.

The Egyptians performed multiplication and division by a method of successive doubling and halving. For example, to multiply 209 by 37 the first number would be successively doubled,

$$\begin{aligned} 209 &= 1\times 209^* \\ 418 &= 2\times 209 \\ 836 &= 4\times 209^* \\ 1672 &= 8\times 209 \\ 3344 &= 16\times 209 \\ 6688 &= 32\times 209^* \end{aligned}$$

Now, since $37=1+4+32$, we add the starred items to obtain

$$209+836+6688 = 7733.$$

In brief, $37\times 209=(1+4+32)209=(2^0+2^2+2^5)209$. The similarity between this method and that used by binary computers is remarkable.

The number 209 expressed in binary notation is 11010001. Doubling this number is equivalent to moving the binary point just as multiplying a decimal number by 10 amounts to moving the decimal point. Since 37 has 100101 as its binary representation, we use the location of the ones to write,

$$\begin{aligned} 11010001 &= 1\times 11010001 \\ 1101000100 &= 100\times 11010001 \\ \underline{1101000100000} &= 100000\times 11010001 \\ 1111000110101 & \end{aligned}$$

This sum corresponds to the decimal number 7733.

In division the Egyptians exhibited their knowledge of inverse processes. To divide 469 by 17, the *divisor* would be successively doubled,

$$\begin{aligned} 17 &= 1\times 17^* \\ 34 &= 2\times 17^* \\ 68 &= 4\times 17 \end{aligned}$$

$$136 = 8 \times 17^*$$
$$272 = 16 \times 17^*$$

Then, since $272+136+34+17=459$ which differs from 469 by 10 which is less than 17, we add the starred items to obtain,

$$(1 + 2 + 8 + 16)17 + 10 = 469.$$

Hence $469 \div 17 = 27$ with a remainder of 10.

Summary. There are a number of striking similarities between the mathematics used in ancient times and the mathematics best adapted to electronic computers. These similarities are due to the fact that mathematicians long ago operated under some of the same constraints that apply to computers today. The following table exhibits these ideas in more concise form.

Mathematicians in ancient times	*Electronic computers today*
1. had to work with limited numeration systems.	1. are designed to use simple numeration systems.
2. had to substitute drudgery for sophistication.	2. make up in speed what they lack in sophistication.
3. had to use computational methods which were simple to understand.	3. are limited in the type and number of instructions they can accept.

Because of these similarities it might be a good idea for the numerical analyst to study ancient mathematics with more than historical interest. He may find algorithms which are ideally suited to the modern electronic computer.

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SEQUENCES OF kth POWERS WITH kth POWER PARTIAL SUMS

DAVID A. KLARNER, Humboldt State College, Arcata, California

Problem 448 (this MAGAZINE, May, 1961) can be generalized as follows: find an infinite sequence $\{a_i\}$ of natural numbers such that for fixed n

$$(1) \qquad \sum_{i=1}^{k(n-1)+1} a_i^n = (a_{k(n-1)+1} + 1)^n \qquad k = 1, 2, \cdots.$$

The case for $n=2$ is completely solved by A. F. Dugan (448, this MAGAZINE,

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Summary. There are a number of striking similarities between the mathematics used in ancient times and the mathematics best adapted to electronic computers. These similarities are due to the fact that mathematicians long ago operated under some of the same constraints that apply to computers today. The following table exhibits these ideas in more concise form.

Mathematicians in ancient times	*Electronic computers today*
1. had to work with limited numeration systems.	1. are designed to use simple numeration systems.
2. had to substitute drudgery for sophistication.	2. make up in speed what they lack in sophistication.
3. had to use computational methods which were simple to understand.	3. are limited in the type and number of instructions they can accept.

Because of these similarities it might be a good idea for the numerical analyst to study ancient mathematics with more than historical interest. He may find algorithms which are ideally suited to the modern electronic computer.

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SEQUENCES OF kth POWERS WITH kth POWER PARTIAL SUMS

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Problem 448 (this MAGAZINE, May, 1961) can be generalized as follows: find an infinite sequence $\{a_i\}$ of natural numbers such that for fixed n

$$(1) \qquad \sum_{i=1}^{k(n-1)+1} a_i^n = (a_{k(n-1)+1} + 1)^n \qquad k = 1, 2, \cdots .$$

The case for $n=2$ is completely solved by A. F. Dugan (448, this MAGAZINE, January, 1962.) The case for $n=3$ will be treated here and suggestions made for other n.

When $n=3$, (1) becomes

$$\sum_{i=1}^{2k+1} a_i^3 = (a_{2k+1}+1)^3. \tag{2}$$

The solution grows out of the identity (Dickson, *History of the Theory of Numbers*, vo. I, p. 599):

$$(uv^2)^3+(3u^2v+2uv^2+v^3)^3+(3u^3+3u^2v+2uv^2)^3=(3u^3+3u^2v+2uv^2+v^3)^3. \tag{3}$$

Letting $v=1$ in (3) gives

$$u^3+(3u^2+2u+1)^3+(3u^3+3u^2+2u)^3 = (3u^3+3u^2+2u+1)^3 \tag{4}$$

which is of the form

$$R^3+S^3+T^3 = (T+1)^3. \tag{5}$$

Transposing some terms in (4) we obtain

$$u^3 = (3u^3+3u^2+2u+1)^3-(3u^3+3u^2+2u)^3-(3u^2+2u+1)^3. \tag{6}$$

That is, the cube of every natural number u is the difference of two consecutive cubes less a cube. A solution of

$$a_1^3+a_2^3+a_3^3 = (a_3+1)^3 \tag{7}$$

is clearly obtainable for any u in (4). Then a_3+1, being a natural number, may be used as u in (6) so that $(a_3+1)^3$ is the difference of two consecutive cubes less a cube, say

$$(a_3+1)^3 = (A+1)^3 - A^3 - C^3. \tag{8}$$

Now define $C=a_4$ and $A=a_5$. Substituting from (8) into (7) gives

$$a_1^3+a_2^3+a_3^3 = (a_5+1)^3 - a_5^3 - a_4^3, \tag{9}$$

and hence

$$a_1^3+a_2^3+a_3^3+a_4^3+a_5^3 = (a_5+1)^3. \tag{10}$$

Now a_5+1 can be used as u in (6) just as a_3+1 was to generate a_6 and a_7. In general, substituting $a_{2n+1}+1$ for u in (6) gives the recursion relations

$$a_{2n+2} = 3(a_{2n+1}+1)^2+2(a_{2n+1}+1)+1, \tag{11}$$

$$a_{2n+3} = 3(a_{2n+1}+1)^3+3(a_{2n+1}+1)^2+2(a_{2n+1}+1). \tag{12}$$

These relations reduce algebraically to

$$a_{2n+2} = 3a_{2n+1}^2+8a_{2n+1}+6, \tag{13}$$

$$a_{2n+3} = 3a_{2n+1}^3+12a_{2n+1}^2+17a_{2n+1}+8. \tag{14}$$

The resulting sequence $\{a_i\}$ given by (13) and (14) and

$$a_1 = u, \tag{15}$$

$$a_2 = 3u^2 + 2u + 1, \tag{16}$$

$$a_3 = 3u^3 + 3u^2 + 2u, \tag{17}$$

has in part the following forms:

$$\{a_i\} = \{1, 6, 8, 262, 2448, \cdots\} \quad \text{if } u = 1, \tag{18}$$

$$\{a_i\} = \{2, 17, 40, 5126, 211888, \cdots\} \quad \text{if } u = 2. \tag{19}$$

The solution given here depends on an identity of the form

$$X_1^j + X_2^j + X_3^j + \cdots + X_j^j = X_{j+1}^j \tag{20}$$

with $j=3$. An identity of the form (20) with $j=5$ is known and will serve as a basis for a similar solution of (1) with $n=5$. Since identities of the form (20) are known only for $j=2$, 3, and 5, the method used is presently limited in scope.

ON EXTREMA IN n-VARIABLES

THOMAS E. MOTT, State University College, Fredonia, New York

The problem of finding the relative maximum and minimum values for a function of n (real) variables is as much a problem of linear algebra as it is of analysis. Unfortunately most texts in the former discipline pay but slight attention to it, deeming it perhaps more a problem of analysis than of algebra. On the other hand, most texts in analysis avoid an algebraic approach. (There is of course one important source where an algebraic treatment is to be found; that being the first volume of Cours D'Analyse de L'École Polytechnique by C. Jordan. However, this book is long out of print and very difficult to purchase today.) Even Harris Hancock's exhaustive work ("Theory of Maxima and Minima," Dover, New York) on the subject does not contain an algebraic development of the problem, although it does use algebraic methods in the final analysis. The purpose of this article is to fill this void as well as to point out the difficulties involved in solving the so-called "ambiguous case."

Let $f(x_1, \cdots, x_n)$ be a function of n variables defined in some region in R_n, and assume that the first partial derivatives $f_{x_1}, \cdots, f_{x_n}$ exist in this region. Necessary conditions for a point $(a_1, \cdots, a_n)$ to be a relative maximum or minimum of $f(x_1, \cdots, x_n)$ are of course that $f_{x_i}(a_1, \cdots, a_n)=0$ $(i=1, \cdots, n)$. With this condition satisfied we now wish to obtain sufficient conditions for $(a_1, \cdots, a_n)$ to be a relative maximum or minimum.

Let us now assume that all the partial derivatives of orders three and less exist and are bounded in some neighborhood of $(a_1, \cdots, a_n)$. Then it follows

The resulting sequence $\{a_i\}$ given by (13) and (14) and

$$a_1 = u, \tag{15}$$

$$a_2 = 3u^2 + 2u + 1, \tag{16}$$

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Let $f(x_1, \cdots, x_n)$ be a function of n variables defined in some region in R_n, and assume that the first partial derivatives $f_{x_1}, \cdots, f_{x_n}$ exist in this region. Necessary conditions for a point $(a_1, \cdots, a_n)$ to be a relative maximum or minimum of $f(x_1, \cdots, x_n)$ are of course that $f_{x_i}(a_1, \cdots, a_n)=0$ $(i=1, \cdots, n)$. With this condition satisfied we now wish to obtain sufficient conditions for $(a_1, \cdots, a_n)$ to be a relative maximum or minimum.

Let us now assume that all the partial derivatives of orders three and less exist and are bounded in some neighborhood of $(a_1, \cdots, a_n)$. Then it follows from Taylor's Theorem that

$$\begin{aligned}\Delta f(h_1, \cdots, h_n) &= f(a_1 + h, \cdots, a_n + h_n) - f(a_1, \cdots, a_n)\\ &= Q(h_1, \cdots, h_n) + G_\theta(h_1, \cdots, h_n),\end{aligned}$$

where

$$Q(h_1, \cdots, h_n) = \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} f_{x_ix_j}(a_1, \cdots, a_n)h_i\cdot h_j,$$

$$G_\theta(h_1, \cdots, h_n) = \frac{1}{3!}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f_{x_ix_jx_k}(a_1 + \theta\cdot h_1, \cdots, a_n + \theta\cdot h_n)h_i\cdot h_j\cdot h_k$$

and $0<\theta<1$ (θ depends on $h_1, \cdots, h_n$).

Now consider the effect of a change of variables $h_j = \sum_{i=1}^n p_{i,j}h_i^*$ ($j=1, \cdots, n$) on the expression $\Delta f(h_1, \cdots, h_n) = Q(h_1, \cdots, h_n) + G_\theta(h_1, \cdots, h_n)$. Let $P = \|p_{ij}\|$; then this may be written in matrix form as $X = (h_1, \cdots, h_n) = (p_{1,1}h_1^* + \cdots + p_{n,1}h_n^*, \cdots, p_{1,n}h_1^* + \cdots + p_{n,n}h_n^*) = X^*P$ where $X^* = (h_1^*, \cdots, h_n^*)$. If $|P| \neq 0$ the matrix P is nonsingular and we also have the inverse representation $(h_1^*, \cdots, h_n^*) = X^* = XP^{-1}$. Thus it follows that every cube $-\epsilon' < h_i^* < \epsilon'$ ($i = 1, \cdots, n$), $\epsilon' > 0$, contains a cube $-\delta' < h_i < \delta'$ ($i=1, \cdots, n$), $\delta'(\epsilon')>0$; hence every sphere $r^* = \sqrt{\{(h_1^*)^2 + \cdots + (h_n^*)^2\}} < \epsilon$, $\epsilon>0$, contains a sphere $r = \sqrt{\{h_1^2 + \cdots + h_n^2\}} < \delta$, $\delta(\epsilon)>0$. Therefore we will be able to consider the extremum problem in terms of $h_1^*, \cdots, h_n^*$. In order to do so we seek a change of variables which reduces the quadratic form $Q(h_1, \cdots, h_n)$ to canonical form.

We begin by writing the quadratic form $Q(h_1, \cdots, h_n)$ as a product of matrices $Q(h_1, \cdots, h_n) = XSX'$ where S is a symmetric matrix. Let $V_n(R)$ be the vector space of all n-tuples $(r_1, \cdots, r_n)$ of real numbers, and $\alpha_1, \cdots, \alpha_n$ be a basis for this n-dimensional space. The quadratic form $Q(h_1, \cdots, h_n)$ may now be considered as a function $Q(\xi) = XSX'$ of the vector $\xi = h_1\alpha_1 + \cdots + h_n\alpha_n$. Let $\alpha_i^* = p_{i,1}\alpha_1 + \cdots + p_{i,n}\alpha_n$ ($i=1, \cdots, n$); then P being a nonsingular matrix, these vectors are linearly independent and form a basis for $V_n(R)$. Furthermore

$$\begin{aligned}h_1^*\alpha_1^* + \cdots + h_n^*\alpha_n^* &= (p_{1,1}h_1^* + \cdots + p_{n,1}h_n^*)\alpha_1 + \cdots\\ &\quad + (p_{n,1}h_1^* + \cdots + p_{n,n}h_n^*)\alpha_n = h_1\cdot\alpha_1 + \cdots + h_n\alpha_n = \xi.\end{aligned}$$

Thus we now see that $h_1^*, \cdots, h_n^*$ are the coordinates of the vector ξ with respect to the new basis $\alpha_1^*, \cdots, \alpha_n^*$. Letting $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\alpha^* = (\alpha_1^*, \cdots, \alpha_n^*)$ we may now write $\alpha^* = \alpha P'$, $\xi = X\alpha' = X^*(\alpha^*)'$, and $X = X^*P$. Finally we note since $\alpha_1, \cdots, \alpha_n$ and $\alpha_1^*, \cdots, \alpha_n^*$ are linearly independent that $\xi=0$ if and only if $h_i=0$ ($i=1, \cdots, n$) and $\xi=0$ if and only if $h_i^*=0$ ($i=1, \cdots, n$). Thus $h_i=0$ ($i=1, \cdots, n$) if and only if $h_i^*=0$ ($i=1, \cdots, n$).

Now $Q(\xi) = Q(h_1, \cdots, h_n) = XSX' = X^*PSP'(X^*)'$. Thus the effect of the change of variables (caused by the change of basis) on the quadratic form $Q(h_1, \cdots, h_n) = XSX'$ with matrix S, is to express it as $Q^*(h_1^*, \cdots, h_n^*) = X^*PSP'(X^*)'$, a quadratic form in $h_1^*, \cdots, h_n^*$ with matrix PSP' congruent to S. Similarly the cubic form $G_\theta(h_1, \cdots, h_n)$ is expressed as a new cubic form $G_\theta^*(h_1^*, \cdots, h_n^*)$.

Let us now consider the general quadratic form $XSX' = \sum_{i=1}^n \sum_{j=1}^n X_i S_{ij} X_j$ in n variables $x_1, \cdots, x_n$, where $X = (x_1, \cdots, x_n)$ and $S = \|S_{ij}\|$ is a symmetric matrix. This form may be reduced by a nonsingular linear transformation $X = ZP$ to the form $z_1^2 + \cdots + z_p^2 - z_{p+1}^2 - \cdots - z_r^2 = ZPSP'Z'$, where r is the rank of matrix S. Let the signature of the form be defined as $s = p - (r-p) = 2p - r$, then r, p, and s are invariants of the form in the sense that they depend only on the form and not on the method used to reduce it.

We now return to the problem concerning the behavior of $f(x_1, \cdots, x_n)$ at (and in a neighborhood of) the point $(a_1, \cdots, a_n)$. If $Q^*(h_1^*, \cdots, h_n^*) \neq 0$ we may write

$$\Delta f(h_1, \cdots, h_n) = Q^*(h_1^*, \cdots, h_n^*) \cdot \left[1 + \frac{G_\theta(h_1^*, \cdots, h_n^*)}{Q^*(h_1^*, \cdots, h_n^*)} \right].$$

We distinguish three cases.

Case I: $r = n$ and either $p = n$ or $p = 0$.

In this case we have $Q^*(h_1^*, \cdots, h_n^*) = \pm [(h_1^*)^2 + \cdots + (h_n^*)^2]$. Now the cubic form $G_\theta^*(h_1^*, \cdots, h_n^*)$ contains three distinct types of terms, terms of the form $(h_i^*)^3$, $(h_i^*)^2 h_j^*$, and $h_i^* h_j^* h_k^*$. Since

$$\frac{(h_i^*)^3}{(h_1^*)^2 + \cdots + (h_n^*)^2} \leq |h_i^*|, \qquad \frac{|(h_i^*)^2 h_j^*|}{(h_1^*)^2 + \cdots + (h_n^*)^2} \leq |h_j^*|,$$

and

$$\frac{|h_i^* \cdot h_j^* \cdot h_k^*|}{(h_1^*)^2 + \cdots + (h_n^*)^2} \leq \frac{\frac{1}{2}[(h_i^*)^2 + (h_j^*)^2]\, |h_k^*|}{(h_1^*)^2 + \cdots + (h_n^*)^2} \leq \frac{|h_k^*|}{2};$$

it follows that

$$\frac{G_\theta^*(h_1^*, \cdots, h_n^*)}{Q(h_1^*, \cdots, h_n^*)} \to 0 \quad \text{as} \quad h_i^* \to 0 \qquad (i = 1, \cdots, n)$$

because the coefficients in $G_\theta^*(h_1^*, \cdots, h_n^*)$ are bounded in some neighborhood of $(a_1, \cdots, a_n)$. Thus there exists $\epsilon^* > 0$ such that $\Delta f(h_1, \cdots, h_n) > 0$ (with $p = n$), $\Delta f(h_1, \cdots, h_n) < 0$ (with $p = 0$) whenever $0 < r^* < \epsilon^*$; and consequently there exists $\epsilon > 0$ such that $\Delta f(h_1, \cdots, h_n) > 0$ (with $p = n$), $\Delta f(h_1, \cdots, h_n) < 0$ (with $p = 0$) whenever $0 < r < \epsilon$.

Case II: $1 \leq p < n$ and $1 \leq r - p < n$, hence $r \geq 2$.

In this case we have $Q^*(h_1^*, \cdots, h_n^*) = (h_1^*)^2 + \cdots + (h_p^*)^2 - (h_{p+1}^*)^2 - \cdots - (h_r^*)^2$. Let $h_i^* = 0$ $(i = p+1, \cdots, n)$, then

$$\frac{G_\theta^*(h_1^*, \cdots, h_p^*, 0, \cdots, 0)}{Q^*(h_1^*, \cdots, h_p^*, 0, \cdots, 0)} \to 0 \quad \text{as} \quad h_i^* \to 0 \qquad (i = 1, \cdots, p).$$

since the coefficients of $G_\theta^*(h_1^*, \cdots, h_n^*)$ are bounded in some neighborhood of $(a_1, \cdots, a_n)$. Thus there exists $\epsilon^*>0$ such that $\Delta f(h_1, \cdots, h_n)>0$ whenever $h_i^*=0$ $(i=p+1, \cdots, n)$ and $0<r^*<\epsilon^*$. On the other hand if we let $h_i^*=0$ $(i=1, \cdots, p, r+1, \cdots, n)$, then

$$\frac{G_\theta^*(0, \cdots, 0, h_{p+1}^*, \cdots, h_r^*, 0, \cdots, 0)}{Q^*(0, \cdots, 0, h_{p+1}^*, \cdots, h_r^*, 0, \cdots, 0)} \to 0 \text{ as } h_i^* \to 0 \qquad (i = p+1, \cdots, r)$$

since the coefficients of $G_\theta^*(h_1^*, \cdots, h_n^*)$ are bounded in some neighborhood of $(a_1, \cdots, a_n)$. Thus there exists $\epsilon^*>0$ such that $\Delta f(h_1, \cdots, h_n)<0$ whenever $h_i^*=0$ $(i=1, \cdots, p, r+1, \cdots, n)$ and $0<r^*<\epsilon^*$.

Case III: $r<n$ and either $p=0$, $r-p=0$, or both (trivial case $r=0$).

In this case we cannot guarantee that

$$\frac{G_\theta^*(h_1^*, \cdots, h_n^*)}{Q^*(h_1^*, \cdots, h_n^*)}$$

tends to zero as $h_i^*\to 0$ $(i=1, \cdots, n)$, in fact we even have $Q^*(h_1^*, \cdots, h_n^*)=0$ for some values of $(h_1^*, \cdots, h_n^*)$ under consideration; hence it becomes necessary to consider forms of order higher than two. This of course requires certain obvious assumptions regarding the existence, boundedness, or continuity of higher ordered partial derivatives of $f(x_1, \cdots, x_n)$ in a neighborhood of the point $(a_1, \cdots, a_n)$. The situation now becomes extremely complicated; however for a particular function $f(x_1, \cdots, x_n)$ it is sometimes possible to obtain results by the methods outlined here. Since forms of odd order cannot be positive or negative definite it may be convenient to consider forms $(F_{2i}+F_{2i+1})$ where the subscript is the order of the form since such forms may be nonnegative or nonpositive for all $(h_1, \cdots, h_n)$ in a neighborhood of $(0, \cdots, 0)$. Thus we may write

$$\Delta f(h_1, \cdots, h_n) = (F_2 + F_3) + \cdots + (F_{2q} + F_{2q+1}) + F_{2q+2}\left[1 + \frac{G_\theta}{F_{2q+2}}\right]$$

where G_θ is an odd $(2q+3)$ ordered form with variable coefficients which are bounded in a neighborhood of $(a_1, \cdots, a_n)$. If $F_{2q+2}=h_1^{2q+2}+ \cdots +h_n^{2q+2}$ plus other even powered terms, it is positive definite, and if $(F_{2i}+F_{2i+1})$ $(i=1, \cdots, q)$ are all nonnegative in a neighborhood of $(0, \cdots, 0)$, then it follows as before that

$$\frac{G_\theta}{F_{2q+2}} \to 0$$

as $h_i\to 0$ $(i=1, \cdots, n)$ (repeated use of $|xy| \leq \frac{1}{2}(x^2+y^2)$) and consequently $\Delta f(h_1, \cdots, h_n)>0$ in a neighborhood of $(0, \cdots, 0)$. Of course it may be necessary to make different nonsingular linear transformations of the variables on each of the forms $(F_{2i}+F_{2i+1})$ $(i=1, \cdots, q)$ and G_θ/F_{2q+2} in order to determine whether these forms have the desired properties. In fact we might even consider nonlinear transformations for this purpose.

Although the method of procedure indicated above in Case III can only apply in rare instances, the reasons for presenting it here are: (1) to indicate the nature of the complexity of the problem and (2) to give some hint of how one might proceed. Essentially one is thrown on his own resources in this case. Indeed serious consideration should be given to the use of geometric methods here.

In Case I the quadratic form $Q(h_1, \cdots, h_n)$ is positive or negative definite, in Case II it is indefinite, and in Case III it is either positive or negative semi-definite with $0<r<n$ or it is the trivial case $r=0$. Thus we may now state our theorem.

THEOREM. *Let all partial derivatives of orders three and less of* $f(x_1, \cdots, x_n)$ *exist and be bounded in some neighborhood of* $(a_1, \cdots, a_n)$, $f_{x_i}(a_1, \cdots, a_n) = 0$ $(i=1, \cdots, n)$, *and*

$$Q = Q(h_1, \cdots, h_n) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} f_{x_i x_j}(a_1, \cdots, a_n) h_i \cdot h_j.$$

Then $f(a_1, \cdots, a_n)$ *is a maximum (minimum) if the quadratic form* Q *is negative (positive) definite, neither a maximum nor a minimum if it is indefinite, and our conclusion must be based on the behavior of higher ordered forms if it is semi-definite.*

In order to make this theorem work we need a simple means of determining the nature of our quadratic form Q. Briefly, a quadratic form XSX' with symmetric matrix $S=\|a_{ij}\|$ is positive definite if and only if

$$\begin{vmatrix} a_{1,1}, & \cdots, & a_{1,m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{m,1}, & \cdots, & a_{m,m} \end{vmatrix} > 0 \qquad (m = 1, \cdots, n)$$

and negative definite if and only if

$$(-1)^m \begin{vmatrix} a_{1,1}, & \cdots, & a_{1,m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{m,1}, & \cdots, & a_{m,m} \end{vmatrix} > 0 \qquad (m = 1, \cdots, n).$$

The proof of these statements as well as the corresponding statements and proof for semi-definite and indefinite forms are to be found in "Introduction to Linear Algebra" by L. Mirsky.

Since the matrix used in representing the quadratic form must be symmetric it is now necessary to describe the method of obtaining this unique symmetric matrix. If $Q(h_1, \cdots, h_n) = XBX'$, let $S=(B+B')/2$ and $K=(B-B')/2$, then $B=S+K$ where S and K are respectively symmetric and skew symmetric; hence $XBX' = XSX' + XKX' = XSX'$. If $B=S_1+K_1$ with S_1 symmetric and K_1 skew symmetric then $B' = S_1' + K_1' = S_1 - K_1$, $B+B' = 2S_1$, $B-B' = 2K_1$, and consequently $S_1 = S$, $K_1 = K$; hence the symmetric matrix S is unique.

Remark. If the quadratic form Q is positive (negative) definite we may delete the condition that the third order partial derivatives of $f(x_1, \cdots, x_n)$ all exist in a neighborhood of $(a_1, \cdots, a_n)$ if we require that all of the first and second order partial derivatives be continuous there. For we may then write

$\Delta f(h_1, \cdots, h_n) = Q_\theta(h_1, \cdots, h_n) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} f_{x_i x_j}(a_1 + \theta h_1, \cdots, a_n + \theta \cdot h_n) h_i h_j$. Let S_θ and S be the symmetric matrices corresponding to the quadratic forms $Q_\theta(h_1, \cdots, h_n)$ and $Q(h_1, \cdots, h_n)$. Since S is positive (negative) definite and the entries in S_θ are continuous and consequently tend to the corresponding entries of S as $h_i \to 0$ $(i=1, \cdots, n)$, then $Q_\theta(h_1, \cdots, h_n)$ will be positive (negative) for sufficiently small values of $h_1, \cdots, h_n$.

In conclusion it should be noted that although we have confined our attention here to (nonsingular) linear transformations it is possible to obtain results from the use of nonlinear transformations. For instance, in the case $n=2$, the nonlinear transformation $h = r \cos \phi$, $k = r \sin \phi$, $r > 0$, $0 \leq \phi < 2$ yields very satisfactory results.

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ON SOME CONTRADICTIONS IN BOUNDARY VALUE PROBLEMS

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The purpose of this paper is twofold. Some contradictions will be pointed out, and a method of solution will be given which will eliminate the contradictions.

The solution to a nonhomogeneous boundary value problem is frequently obtained as a series of characteristic functions which satisfy homogeneous boundary conditions. The result obtained is then valid in the open region but not the closed region. Since the answer does not satisfy the boundary condition, there is an apparent contradiction. A problem in linear heat conduction will be used to illustrate this matter, and it will be shown how the discrepancy can be eliminated. The key point in the analysis is the use of equation (9) instead of the customary equation (8). The procedure given here is not the only way for going from (9) to (17).

Consider the problem of linear heat flow in a slab with temperature given at one end. That is,

$$(1) \quad \kappa \frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}, \qquad (2) \quad U(0, t) = U(x, 0) = 0, \qquad (3) \quad U(\pi, t) = F(t).$$

From [1, p. 104], one obtains the solution to this problem as

$$(4) \qquad U = \frac{2\kappa}{\pi} \sum_{n=1}^{\infty} n(-1)^{n+1} \sin nx \int_0^t F(\tau) e^{-\kappa n^2 (t-\tau)} d\tau.$$

$\Delta f(h_1, \cdots, h_n) = Q_\theta(h_1, \cdots, h_n) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n f_{x_ix_j}(a_1+\theta h_1, \cdots, a_n+\theta \cdot h_n)h_ih_j$. Let S_θ and S be the symmetric matrices corresponding to the quadratic forms $Q_\theta(h_1, \cdots, h_n)$ and $Q(h_1, \cdots, h_n)$. Since S is positive (negative) definite and the entries in S_θ are continuous and consequently tend to the corresponding entries of S as $h_i \to 0$ $(i=1, \cdots, n)$, then $Q_\theta(h_1, \cdots, h_n)$ will be positive (negative) for sufficiently small values of $h_1, \cdots, h_n$.

In conclusion it should be noted that although we have confined our attention here to (nonsingular) linear transformations it is possible to obtain results from the use of nonlinear transformations. For instance, in the case $n=2$, the nonlinear transformation $h=r \cos \phi$, $k=r \sin \phi$, $r>0$, $0 \leq \phi < 2$ yields very satisfactory results.

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ON SOME CONTRADICTIONS IN BOUNDARY VALUE PROBLEMS

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The purpose of this paper is twofold. Some contradictions will be pointed out, and a method of solution will be given which will eliminate the contradictions.

The solution to a nonhomogeneous boundary value problem is frequently obtained as a series of characteristic functions which satisfy homogeneous boundary conditions. The result obtained is then valid in the open region but not the closed region. Since the answer does not satisfy the boundary condition, there is an apparent contradiction. A problem in linear heat conduction will be used to illustrate this matter, and it will be shown how the discrepancy can be eliminated. The key point in the analysis is the use of equation (9) instead of the customary equation (8). The procedure given here is not the only way for going from (9) to (17).

Consider the problem of linear heat flow in a slab with temperature given at one end. That is,

$$(1)\quad \kappa \frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}, \qquad (2)\quad U(0, t) = U(x, 0) = 0, \qquad (3)\quad U(\pi, t) = F(t).$$

From [1, p. 104], one obtains the solution to this problem as

$$(4)\qquad U = \frac{2\kappa}{\pi} \sum_{n=1}^{\infty} n(-1)^{n+1} \sin nx \int_0^t F(\tau) e^{-\kappa n^2(t-\tau)} d\tau.$$

It is seen at once that (4) does not satisfy the nonhomogeneous boundary condition (3). Moreover, if one formally differentiates the series termwise, it does not appear to satisfy the differential equation either. These discrepancies are well known, but are frequently not mentioned.

The problem specified by (1)–(3) is not solved by the standard separation of variable procedure. Carlsaw and Jaeger [1] obtained (4) by means of Duhamel's theorem. The Laplace transform, or Green's function [1, chapter XIV], might equally well have been used. An alternate procedure will now be presented.

The characteristic functions, $X_n(x)$, for this problem satisfy the equations

$$\frac{d^2X_n}{dx^2} = -\xi_n^2 X_n, \tag{5}$$

$$X_n(0) = X_n(\pi) = 0. \tag{6}$$

A solution to this system is

$$X_n = \sin nx, \tag{7}$$

where n is an integer. Any constant times this is also a solution.

If the separation of variable procedure were tried, one would postulate a solution of the form

$$U = \sum_1^\infty A_n(t) \sin nx. \tag{8}$$

Instead of this, let us put

$$\sum_1^\infty B_n(t) \sin nx = \frac{1}{\kappa}\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}\cdot \tag{9}$$

Now use the orthogonality of the functions sin nx to obtain the coefficient $B_n(t)$ in the usual way, and then integrate the term in $\partial^2 U/\partial x^2$ twice by parts. This gives

$$\pi B_n(t)/2 = \frac{1}{\kappa}\int_0^\pi \sin nx \frac{\partial U}{\partial t}\,dx \tag{10}$$

$$= \int_0^\pi \sin nx \frac{\partial^2 U}{\partial x^2}\,dx \tag{11}$$

$$= n(-1)^{n+1}F(t) - n^2\int_0^\pi U \sin nx dx. \tag{12}$$

From (10) and (12):

$$\kappa n(-1)^{n+1}F(t) = \int_0^\pi \sin nx\left(\frac{\partial U}{\partial t} + \kappa n^2 U\right)dx \tag{13}$$

$$\kappa n(-1)^{n+1}F(t) = e^{-\kappa n^2 t}\int_0^\pi \sin nx \frac{\partial}{\partial t}\,(Ue^{\kappa n^2 t})dx. \tag{14}$$

So

$$\int_0^\pi U \sin nx dx = \kappa n(-1)^{n+1} \int_0^t F(\tau) e^{-\kappa n^2(t-\tau)} d\tau. \tag{15}$$

Next integrate (9) twice with respect to x from 0 to x. The answer will be in terms of $U_x(0, t)$, which can be eliminated by using (3). The result is

$$U = xF(t)/\pi - \sum_1^\infty \frac{B_n(t)}{n^2} \sin nx. \tag{16}$$

But $B_n(t)$ is known from (12) and (15), so

$$\begin{aligned} U = {} & \frac{F(t)}{\pi}\left[x - 2\sum_1^\infty \frac{(-1)^{n+1}}{n} \sin nx\right] \\ & + \frac{2\kappa}{\pi} \sum_1^\infty (-1)^{n+1} n \sin nx \int_0^t F(\tau) e^{-\kappa n^2(t-\tau)} d\tau. \end{aligned} \tag{17}$$

This now is the solution to (1)–(3) obtained by means of (9). Observe first that it satisfies (3). Let

$$\pi S(x) = x - 2\sum_1^\infty \frac{(-1)^{n+1}}{n} \sin nx. \tag{18}$$

Then it is seen that the result (17) differs from (4) in that (17) has the extra term $F(t)S(x)$. However, the Fourier sine series for x in the interval $(-\pi, \pi)$ is

$$x = 2\sum_1^\infty \frac{(-1)^{n+1}}{n} \sin nx \qquad \text{for } -\pi < x < \pi. \tag{19}$$

But this series does not represent x outside the interval $(-\pi, \pi)$ or even at the end points. At $x=\pi$ it gives zero instead of π. Hence

$$\begin{cases} S(x) = 0 & \text{for } 0 \leqq x < \pi, \\ S(x) = 1 & \text{for } \quad x = \pi. \end{cases} \tag{20}$$

Thus (17) differs from (4) by a function which is zero everywhere within the range $0 \leqq x < \pi$, and which is equal to $F(t)$ at $x=\pi$. Or, more briefly, the two solutions differ in the range of interest only by a null function.

Next observe that (17) gives

$$U(x, 0) = F(0)S(x), \tag{21}$$

which is zero except at $x=\pi$, where it is equal to $F(0)$. Thus (17) satisfies (2) in the open region $0 \leqq x < \pi$. However, (2) and (3) do not agree on the value of $U(\pi, 0)$. The solution (17) gives the value prescribed by (3) rather than that implied by (2).

Finally, substitute (17) into (1), and formally differentiate the series termwise. The result is

$$\kappa \frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial t} = - F'(t)S(x). \tag{22}$$

Hence the differential equation is formally satisfied except at $x=\pi$, where one finds $\partial^2 U/\partial x^2 = 0$ and $\partial U/\partial t = F'(t)$.

In summary then, one must realize that (1)–(3) are not in complete agreement. Where they agree, (17) is found to hold. Where there is a disagreement, (17) gives one or the other of the values specified. Also, (4) and (17) give the same results except at $x=\pi$, where the temperature is known. So, unless one is concerned about the contradiction at $x=\pi$, (4) is suitable for determining $U(x, t)$.

The initial temperature of the region was specified as zero for simplicity. However, if $U(x, 0)$ is some prescribed function of x, then this appears without difficulty in (15), and hence (17). On the other hand, if one were solving by means of the Laplace transform, an arbitrary initial temperature would introduce considerable additional effort, as may be seen by [2, p. 219].

If one takes the Laplace transform of (1)–(3), and solves for the transform of $U(x, t)$, he obtains

$$u(x, s) = \frac{f(s) \sinh x\sqrt{(s/\kappa)}}{\sinh \pi\sqrt{(s/\kappa)}} \cdot \tag{23}$$

The inverse transform of this may be obtained by means of the convolution integral. If in doing so, $f(s)$ is taken as one function, and the rest of the right side of (23) as the second, then the result given in (4) is obtained. However, Churchill [2, p. 212] has multiplied the numerator and denominator of (23) by s, and used $sf(s)$ as the first function. Upon taking inverse transforms with this choice, one obtains a result which, when integrated once by parts, gives (17). Thus the inverse transform of (23) can be taken in two slightly different ways, one of which leads to (4), and the other to (17). It has been shown, however, that these two results differ only by a null function in the interval 0 to π.

For further discussion of the contradiction in the above problem, one may see [3]. A similar difficulty in another boundary value problem is considered in [4].

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THE OCCURRENCE OF DIGITS

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Let a, x, and b be integers such that $x>1$, $b>1$, and $0\leqq a\leqq b-1$. Let $p(a, x) = p(a, x, b)$ denote the number of occurrences of the digit a in the list of integers from and including zero up to but not including x, each written to the base b so that every integer in this list has as many digits as does $x-1$ (by introducing, if necessary, antecedent 0's).

Suppose $x=b^m$. If $m>0$, each of the b digits occurs equally often. Hence

$$p(a, b^m) = mb^{m-1}. \tag{1}$$

Suppose $x=cb^m$, with $1\leqq c<b$. The first m digits from the right form c copies of the preceding case. For $a\geqq c$, these are the only occurrences of a. For $a<c$, a will occur as the first digit from the left exactly b^m times. Thus

$$p(a, cb^m) = \begin{cases} cmb^{m-1} & \text{if } a \geqq c \\ cmb^{m-1} + b^m & \text{if } a < c. \end{cases} \tag{2}$$

For $c=1$, and $a\geqq c$, (2) reduces to (1). In fact (2) is valid for all c and a except $c=1$, $a=0$, in which case $p(a, cb^m)$ is too large by b^m.

More generally, let $x=\sum_{i=0}^n c_i b^i$, where $0\leqq c_i<b$ for all i and $c_n\neq 0$. For $m=0, 1, \cdots, n$, let $x_m=\sum_{i=m}^n c_i b^i$. Then $x_0=x$ and $x_n=c_n b^n$. For $m=0, 1, \cdots, n-1$, let

$$\begin{aligned} q(a, m) &= p(a, x_m) - p(a, x_{m+1}) \\ &= \text{the number of occurrences of } a \text{ in the list of integers from } x_{m+1} \text{ up} \\ &\quad\ \text{to but not including } x_m. \end{aligned} \tag{3}$$

The number of occurrences of a among the first $m+1$ digits from the right is $p(a, c_m b^m)$. The number of occurrences of a among the first remaining digits from the left is

$$\begin{aligned} &c_m b^m \times (\text{the number of } c_i\text{'s that are } a, \text{ for } m+1 \leqq i \leqq n) \\ &\quad = c_m b^m \sum_{i=m+1}^{n} \delta(a, c_i), \end{aligned}$$

where δ is the Kronecker delta function. Thus

$$q(a, m) = p(a, c_m b^m) + c_m b^m \sum_{i=m+1}^{n} \delta(a, c_i). \tag{4}$$

For the first term to the right of (4) we can use (2), even if $c_m=0$. The warning given regarding the use of (2) for $c=1$ and $a=0$ can be safely ignored (since $m<n$ so 0 will occur b^m times as the $(m+1)$-st digit from the right) in every case except that for which $m=n-1$, $c_n=1$, and $c_i=0$ for $0\leqq i\leqq n-1$. In this case (2) will give a result too large by b^n for $a=0$. We shall see later that even this warning can be ignored.

From (3) and (4) we have

$$p(a, x) = \sum_{m=0}^{n-1} q(a, m) + p(a, x_n) \tag{5}$$
$$= \sum_{m=0}^{n} p(a, c_m b^m) + \sum_{m=0}^{n-1} c_m b^m \sum_{i=m+1}^{n} \delta(a, c_i).$$

Inserting (2) into the first term and changing the order of summation of the second term of (5), we have:

$$p(a, x) = \sum_{m=1}^{n} c_m m b^{m-1} + \sum{}^{*} b^m + \sum{}^{**} \sum_{m=0}^{i-1} c_m b^m, \tag{6}$$

where $\sum^{*} b^m$ indicates we have a term of b^m for every m, $0 \leqq m \leqq n$, such that $c_m > a$, and $\sum^{**} \sum_{m=0}^{i-1} c_m b^m$ indicates we have a term of $\sum_{m=0}^{i=1} c_m b^m$ for every i, $1 \leqq i \leqq n$, such that $c_i = a$. As noted before (6) is valid everywhere except $c_n = 1$ and $c_i = 0$ for $0 \leqq i \leqq n-1$, in which case $p(0, x)$ is too large by b^n.

Let us redefine $p(a, x)$, denoted by $p'(a, x)$, as the number of occurrences of the digit a in the list of integers from *one* up to and not including x, each written to the base b (and so dropping the requirement that each integer have as many digits as does $x-1$). Only $p(0, x)$ is changed. For this we must subtract from $p(0, x)$ the quantity

$$\sum_{m=0}^{n} b^m = \frac{b^{n+1} - 1}{b - 1} \tag{7}$$

unless $c_n = 1$ and $c_i = 0$ for $0 \leqq i \leqq n-1$, in which case (7) is too large by b^n. But in this case formula (6) for $p(0, x)$ is too large by b^n. Therefore, in any case,

$$p'(0, x) = p(0, x) - \frac{b^{n+1} - 1}{b - 1}, \tag{8}$$

where $p(0, x)$ is given by (6).

LOANS WITH A PARTIAL PAYMENT

HUGH E. STELSON, Michigan State University

Introduction. In many loans a specified payment is required. Such loans are likely to have a final partial payment. That is, for a loan of B, n full payments of R each are made at the end of each period for n periods and a final payment, $fR(0 < f < 1)$ is made after $n+1$ periods. The final partial payment may or may not be given. Usually if B and the rate r are given then fR is not given. This applies particularly to mortgages. On the other hand, in many small loans fR is given but r is not given.

From (3) and (4) we have

$$(5)\qquad \begin{aligned} p(a, x) &= \sum_{m=0}^{n-1} q(a, m) + p(a, x_n) \\ &= \sum_{m=0}^{n} p(a, c_m b^m) + \sum_{m=0}^{n-1} c_m b^m \sum_{i=m+1}^{n} \delta(a, c_i). \end{aligned}$$

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1. Computation of the Partial Payment. Let B be the original debt, I the total cost of the loan, and R the specified payment. Then, if $B+I$, the total repayment, is divided by R we obtain $n+f$ where n is an integer and $0<f<1$. Hence Rf is the last partial payment. That is,

$$f = \frac{B+I}{R} - n. \tag{1}$$

Now (1) is a useful formula only if I is given. In case I is not given, we set up the formula $B=Ra_{\overline{k}|}$ and use an annuity table to find an integer, n. Then we have

$$f = (\theta - a_{\overline{n}|})(1+r)^{n+1}, \quad \text{where} \quad \theta = \frac{B}{R} \tag{2}$$

or $f=(\theta-a_{\overline{n}|})/(a_{\overline{n+1}|}-a_{\overline{n}|})$ which shows that f can be found by interpolation in the $a_{\overline{n}|}$ table. Again, if n is known, f can be obtained from the formula $\log(1-frv) =\log(1-r\theta)+n\log(1+r)$.

To determine f for a specified payment with a given add-on charge, we proceed as follows: Let the specified add-on charge be c per period, then $Rn =B(1+cn)$ or

$$n + f = \frac{B}{R - Bc} = \frac{\theta}{1 - \theta c} \quad \text{where} \quad \theta = \frac{B}{R}\cdot$$

To find f for a loan with a graduated interest rate, $r\%/s\%/L/B$, we can use the formula

$$B + (L(r-s) + Bs - R)s_{\overline{m}|(s)} = Ra_{\overline{n-m}|(r)} + fRv^{n-m+1}$$

to solve for f, where m the breakpoint and n (an integer) are first determined [1].

Let us now compute f without the use of tables. If we take the equation,

$$\theta = a_{\overline{k}|} \text{ at } (r) \quad \text{where} \quad \theta = \frac{B}{R} \tag{3}$$

and solve for k, we have

$$k = \frac{-\ln(1-\theta r)}{\ln(1-r)} = \frac{2\left[x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \cdots\right]}{2\left[y + \dfrac{y^3}{3} + \dfrac{y^5}{5} + \cdots\right]}, \quad x = \frac{\theta r}{(2-\theta r)} \text{ and } y = \frac{r}{(2+r)}.$$

Consider the numerator of k, namely,

$$2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

This can be expressed as a continued fraction [5]

$$2x\left[\frac{1|}{|1}-\frac{x^2|}{|3}-\frac{4x^2|}{|5}-\frac{9x^2|}{|7}-\frac{16x^2|}{|9}-\cdots\right].$$

The second and fifth approximants are, respectively,

$$X_2 = 2x(15 - 4x^2)/(15 - 9x^2) \text{ and}$$

$$X_5 = 2x(1155 - 1190x^2 + 231x^4)/(1155 - 1575x^2 + 525x^4 - 25x^6).$$

Therefore

$$(4) \qquad k = \frac{X_2}{Y_2} \text{ or } \qquad (5) \qquad k = \frac{X_5}{Y_5}$$

where Y_2 and Y_5 are the corresponding approximants for the denominator. Now k is an approximation for the time k in (1). What is wanted is the time in the equation

$$(6) \qquad \theta = a_{\overline{n}|} + fv^{n+1}.$$

The time as given by (3) is the same as in (1) only if k is an integer. Otherwise the time by (3) corresponds to simple interest for the fractional period; whereas the time given by (1) gives the result for compound interest [4]. Let $k=n+g$, then $g<f$ since compound interest is less than simple interest for fractional periods. If g is first determined, then $f=2g-s_{\overline{g}|}=g+E$, where the error E is given by $E=g-s_{\overline{g}|}(r)$.

$$(7) \qquad E = \frac{(1-g)gr}{2} - \frac{(1-g)(2-g)gr^2}{6} + \cdots$$

Example I. (Mortgage). A loan of \$10,000 is repaid by monthly payments of \$100. If the rate is 6% compounded monthly, find the number of full payments and the last partial payment.

Solution A (Annuity tables).

Since $10{,}000=100\, a_{\overline{k}|}$ at $(\frac{1}{2})\%$ we find in a present value annuity table that $n=138+$. Using the formula (2) the final partial payment is $f=(100-a_{\overline{138}|})\cdot(1.005)^{139}=97.578065$.

Solution B (Formula).

By formula (4) where

$$X_2 = \frac{2}{3}\left(15-\frac{4}{9}\right)\Big/ 14 = .69312169 \text{ and}$$

$$Y_2 = \frac{2}{401}\left(15-\frac{4}{401}\right)\Big/\left(15-\frac{9}{401}\right) = .00499168$$

we have $k=X_2/Y_2=138.855$.

By formula (5) $k=X_5/Y_5=138.97570$ where $X_5=.6931471$ and $Y_5=.00498754$. This is within 1% of the correct result. Using one term of formula (7) we obtain $E=.00006$ which added to g gives the correct f.

2. Elimination of the Partial Payment. The last partial payment might well be eliminated in one of two ways.

(a) The payment R, for the first n periods might be increased to $R(1+\alpha)$ so as to pay off the obligation. With this condition in mind, we set up the equation,

$$B = Ra_{\overline{n}|} \text{ at } (r) + fRv^{n+1} = R(1 + \alpha)a_{\overline{n}|} \text{ at } (r). \tag{8}$$

Solving for α gives

$$\alpha = f/(1 + r)s_{\overline{n}|} \text{ at } (r) \tag{9}$$

which can be used without knowing B. Since $B=R(1+\alpha)a_{\overline{n}|}$, we can eliminate n from (9) and obtain

$$\alpha = \frac{f(R - Br)}{B(1 + r) - fR},$$

a result not containing n. Again, from (8) we find

$$\alpha = \frac{B}{Ra_{\overline{n}|}} - 1, \tag{10}$$

so that α can be obtained without f.

(b) The payment R might be decreased to $R(1-\alpha)$ so as to pay off the obligation in $n+1$ payments. Then $B=Ra_{\overline{n}|}+fRv^{n+1}=R(1-\alpha)a_{\overline{n+1}|}$ and

$$\alpha = \frac{1 - f}{s_{\overline{n+1}|}} \tag{11}$$

or

$$\alpha = 1 - \frac{B}{Ra_{\overline{n+1}|}}\cdot \tag{12}$$

From Example I it was found that 138 full payments were required plus a payment of \$97.578065 for the 139th month. Thus $f=.97578065$. By (9) this means that if each regular payment is increased by 49¢ the partial payment is annulled. Again, if the payments were reduced by (11) to \$99.99 the loan would be repaid in exactly 139 payments.

3. Finding the Interest Rate. In many small loans the payments are specified (including the partial payment) but the interest rate is not stated. In such cases it may be desirable to find the interest rate by formulas such that the result is sufficiently accurate for a schedule. The formula is worthwhile since interpolation in tables with a partial payment might be tedious.

We have the relation $B=Ra_{\overline{n}|}+fRv^{n+1}$ to solve for r, the rate of interest. Since $B+I=R(n+f)$, we eliminate R and obtain

$$B = \frac{I(a_{\overline{n}|} + fv^{n+1})}{n + f - (a_{\overline{n}|} + fv^{n+1})},$$

or, expanding,

$$B\left[\left(1+\frac{f}{n+f}\right)\frac{(n+1)r}{2}-\left(1+\frac{2f}{n+f}\right)\frac{(n+1)(n+2)r^2}{6}-\cdots\right]$$
$$=I\left[1-\left(1+\frac{f}{n+f}\right)\frac{(n+1)r}{2}+\left(1+\frac{2f}{n+f}\right)\frac{(n+1)(n+2)r^2}{6}-\cdots\right]$$

or

$$B\left[\left(\frac{n+2f}{n+f}\right)\frac{(n+1)r}{2}\right]=\left[1-\frac{n(n^2+4nf+6f^2-n-4f)r}{6(n+f)(n+2f)}+\cdots\right].$$

Neglecting terms containing r^2 and higher powers, we have

$$r=\frac{6I(n+f)(n+2f)}{3B(n+1)(n+2f)^2+In(n^2-n+4nf+6f^2-4f)},\tag{13}$$

where n is the number of regular payments.

This is the analogue of the Direct Ratio formula since it was obtained in exactly the same manner [3].

The rate obtained as in the case of the Direct Ratio formula is always slightly small.

Example. For a loan of \$88.91 the borrower agrees to pay \$10 at the end of each of ten months and \$5 at the end of the eleventh month. In this case $R=10$, $fR=5$, $I=16.09$, $n=10$, $B=88.91$, and

$$r=\frac{6(16.09)(11)(10.5)}{3(88.91)(11)(11)^2+160.9(90+20-.5)}=2.992\%$$

with an error of .0008.

If we neglect the terms, $2Inf(f-1)$ in the denominator, the factor $n+2f$ can be divided out and we have a slightly smaller value of r given by

$$r=\frac{6I(n+f)}{3B(n+1)(n+2f)+In(n+2f-1)}.\tag{14}$$

For the above example, $r=2.9916$ with an error of .000084. Formula (14) still reduces to the direct ratio formula when $f=0$.

Example. On an unpaid balance of \$50.01 to \$55.00 a mail order company adds \$6.50 for the credit price and requires a payment of \$5 per month.

(a) The time price for a balance of \$50.01 is \$56.51 to be paid at \$5 per month for 11 months and \$1.51 the twelfth month. By (13)

$$r=\frac{6(6.50)(11.302)(11.604)}{150.03(12)(11.604)^2+6.50(11)(122.627)}=2.03\%\text{ per month.}$$

By (14), $r=2.03\%$, the same as (13).

(b) Let the unpaid balance be \$53.50. Then the time price requires 12 payments of \$5 each. Hence

$$r = \frac{2(6.50)}{53.50(13) + \dfrac{6.50}{3}(11)} = 1.807\% \text{ per month.}$$

(c) Let the unpaid balance be \$55. Then the time price is \$61.50 which requires 12 payments of \$5 each and a thirteenth payment of \$1.50. By (14)

$$r = \frac{6(6.50)(12.3)}{3(55)(13)(12.6) + 6.50(12)(11.6)} = 1.717\%.$$

These examples point out the variation in the rate as affected by the partial payment.

Example. On an unpaid balance of \$50.01 to \$55.00 a mail order company adds \$6.50 for the credit price and requires a payment of \$5 per month.

By solving an equation of the form $50+x=5a_{\overline{11}|}+(1.50+x)v^{12}$ and by the use of tables we find that on an unpaid balance of \$54.48 the company makes exactly $1\frac{3}{4}\%$ per month, while for an unpaid balance of \$50.55 the company makes exactly 2% per month.

We use these results to check formulas (13) and (14).

Unpaid balance	*Formula* (13)	*Error*	*Formula* (14)	*Error*
50.55	1.9977%	.00023%	1.9969%	.00031%
54.48	1.748%	.0002%	1.748%	.0002%

Thus formula (14) should be sufficiently accurate for use in making a schedule.

References

1. H. E. Stelson, Graduated interest rates in small loans, Amer. Math. Monthly, 69 (1962) 15–21.
2. ———, Mathematics of finance, Van Nostrand, Princeton, N. J., 1957, Appendix 9.
3. ———, The rate of interest in installment payment plans, Amer. Math. Monthly, 60 (1953) 326–329.
4. ———, The accuracy of linear interpolation in tables of the mathematics of finance, Mathematical Tables and Other Aids to Computation, April, 1949.
5. Erich Michalup, Some approximation formulae of the effective rate and the force of interest, Skand. Aktuarietidskrift, 1955, p. 163.

A NOTE ON COEFFICIENTS OF CYCLOTOMIC POLYNOMIALS

HELEN HABERMEHL, SHARON RICHARDSON, AND MARY ANN SZWAJKOS,
Rosary Hill College

In discussing the monic cyclotomic polynomial $F_d(x) = \sum_{n=0}^{\phi(d)} c_n x^n$, it is convenient to have a concise method of obtaining the coefficients, c_n. In this paper we will give a method of finding the coefficient c_n when $d=3p$, p a prime greater than 3.

$$r = \frac{2(6.50)}{53.50(13) + \frac{6.50}{3}(11)} = 1.807\% \text{ per month.}$$

(c) Let the unpaid balance be \$55. Then the time price is \$61.50 which requires 12 payments of \$5 each and a thirteenth payment of \$1.50. By (14)

$$r = \frac{6(6.50)(12.3)}{3(55)(13)(12.6) + 6.50(12)(11.6)} = 1.717\%.$$

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A more general result is stated by Sister Marion Beiter [1, pp. 31–32] for c_n in F_d when d is the product of two distinct odd primes, p_1 and p_2. One result of [1] states that the coefficient $c_n=(-1)^m$ when n has exactly one partition in the form $n=rp_1+sp_2+m$, $r, s, =0, 1, \cdots, m=0, 1$, and is zero otherwise. The result can be put in a much simpler form in the case $d=3p$.

Let $F_{3p}(x)=\sum_{n=0}^{2(p-1)} c_nx^n$ be the monic cyclotomic polynomial whose zeros are the primitive $3p$th roots of unity.

THEOREM. *If F_{3p} is defined as above, then for $n\leqq p-1$,*

$$c_n = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3 \\ -1 & \text{if } n \equiv 1 \pmod 3 \\ 0 & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

for $n>p-1$, $c_n=c_{n'}$, where $n'=2(p-1)-n$.

This theorem follows immediately from the concept of a modular system and the theorem on partitions as given by [1]. We chose to prove the theorem directly from the above definition of cyclotomic polynomial.

Proof. From [2, pp. 113–114] we have

$$x^{3p} - 1 = \prod_{h/3p} F_h(x),$$

where h runs over the positive divisors of $3p$. Therefore

$$\begin{aligned} x^{3p} - 1 &= F_1(x)F_3(x)F_p(x)F_{3p}(x) \\ &= (x-1)(x^{p-1}+x^{p-2}+\cdots+x+1)(x^2+x+1)F_{3p}(x). \end{aligned}$$

Then $(x^p-1)(x^2+x+1)F_{3p}(x)=x^{3p}-1$. Since

$$F_{3p}(x) = \sum_{n=0}^{2(p-1)} c_nx^n,$$

we have

$$(x^2+x+1)\cdot \sum_{n=0}^{2(p-1)} c_nx^n = x^{2p}+x^p+1.$$

$$\begin{aligned} (x^2+x+1)\cdot \sum_{n=0}^{2(p-1)} c_nx^n &= \sum_{n=0}^{2(p-1)} c_nx^{n+2} + \sum_{n=0}^{2(p-1)} c_nx^{n+1} + \sum_{n=0}^{2(p-1)} c_nx^n \\ &= \sum_{n=2}^{2p} c_{n-2}x^n + \sum_{n=1}^{2p-1} c_{n-1}x^n + \sum_{n=0}^{2(p-1)} c_nx^n. \end{aligned}$$

So

$$\sum_{n=0}^{2(p-1)} c_nx^n + \sum_{n=1}^{2p-1} c_{n-1}x^n + \sum_{n=2}^{2p} c_{n-2}x^n = 1 + x^p + x^{2p}.$$

Since this is an identity, equating coefficients of like powers of x from both sides of the identity gives:

$$
\begin{aligned}
&x^0 : c_0 = 1 && \\
&x^1 : c_1 + c_0 = 0, && c_1 = -1 \\
&x^2 : c_2 + c_1 + c_0 = 0, && c_2 = 0 = -c_1 - c_0 \\
&x^3 : c_3 + c_2 + c_1 = 0, && c_3 = 1 = -c_2 - c_1 \\
&\cdots\cdots\cdots\cdots && \\
&x^n : c_n = -c_{n-1} - c_{n-2} && \text{for } 2 \leqq n \leqq p-1.
\end{aligned}
$$

For $n>p-1$, $c_n=c_{2(p-1)-n}$ from the symmetry of the polynomial. Thus the theorem is proved. E.g.,

$$F_{15}(x) = c_0 + c_1x + \cdots + c_7x^7 + c_8x^8$$

$$
\begin{aligned}
&0 \equiv 0 \pmod 3, && c_0 = 1 && \\
&1 \equiv 1 \pmod 3, && c_1 = -1 && c_5 = c_3 = 1 \\
&2 \equiv 2 \pmod 3, && c_2 = 0 && c_6 = c_2 = 0 \\
&3 \equiv 0 \pmod 3, && c_3 = 1 && c_7 = c_1 = -1 \\
&4 \equiv 1 \pmod 3, && c_4 = -1 && c_8 = c_0 = 1
\end{aligned}
$$

$$F_{15}(x) = 1 - x + x^3 - x^4 + x^5 - x^7 + x^8.$$

The authors are indebted to the referee for his suggestions for this paper's improvement.

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1. Sister Marion Beiter, O.S.F., Coefficients in the cyclotomic polynomial for the numbers with at most three distinct odd primes in their factorization, Washington, D. C., 1960.

2. B. L. van der Waerden, Modern Algebra, vol. I New York, 1949.

ON THE RATIONAL CONGRUENCE OF TERNARY QUADRATIC FORMS

AMIN MUWAFI, American University of Beirut

Introduction. Let

$$f = \sum_{i,j=1}^{n} a_{ij}x_ix_j \quad \text{and} \quad g = \sum_{i,j=1}^{n} b_{ij}y_iy_j$$

be two quadratic forms with real coefficients and $a_{ij}=a_{ji}$, $b_{ij}=b_{ji}$ for all i and j. If there is a nonsingular linear transformation, with real coefficients, which takes f into g we call f and g *congruent forms*. If the coefficients are rational the forms are said to be *rationally congruent*. Associated with f is the determinant of the matrix (a_{ij}) whose value will be denoted by d. If A_{ij} denotes the cofactor of a_{ij}

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\end{aligned}
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in $|a_{ij}|$ then the form ϕ defined by

$$\phi = \sum_{i,j=1}^{n} A_{ij}x_ix_j, \qquad A_{ij} = A_{ji},$$

is called the *adjoint* of the form f.

If a ternary quadratic form is transformed into another form, by a linear transformation, the coefficients of the resulting form may be computed by direct substitution (a tedious job), or by matrix multiplication, or by the use of explicit formulas. The lemma, stated below, gives these formulas and will be needed for further development. It is contained in Theorem 2, page 6, of [1]. It is stated here in a different notation for convenience.

LEMMA. *If* (c_{ij}), $i, j=1, 2, 3$ *is the matrix of a linear transformation of determinant* $|c_{ij}| = C \neq 0$, *which takes a ternary quadratic form* f *with coefficients* $a_{ij}=a_{ji}$ *into another form* g, *then the coefficients* b_{ij}, $i, j=1, 2, 3$ *of* g *are given by*

$$(1) \qquad b_{ij} = X_{1j}c_{1i} + X_{2j}c_{2i} + X_{3j}c_{3i}, \qquad b_{ij} = b_{ji},$$

where

$$(2) \qquad X_{ij} = a_{i1}c_{1j} + a_{i2}c_{2j} + a_{i3}c_{3j}, \qquad i, j = 1, 2, 3.$$

In particular, $b_{jj}=f(c_{1j}, c_{2j}, c_{3j})$. *The determinant of* g *is* C^2d.

It is a well-known theorem that a quadratic form f in n variables and rank r, $(r \leqq n)$ can be reduced by a real nonsingular linear transformation to the normal form $g = \sum_{i=1}^{r} b_iy_i^2$, where the b's are nonzero constants. The theorem below, applied to ternary quadratic forms, shows that normalization is possible with a linear transformation whose determinant has a pre-assigned value $C \neq 0$, and whose coefficients are rational numbers.

THEOREM. *Every ternary quadratic form*

$$(3) \qquad f = \sum_{i,j=1}^{3} a_{ij}x_ix_j, \qquad a_{ji} = a_{ij},$$

with determinant $d \neq 0$ *and with rational coefficients, can be taken into the normal form*

$$(4) \qquad g = \sum_{i=1}^{3} b_{ii}y_{ii}^{2},$$

where the b'*s are nonzero rational numbers, by a linear rational transformation* (c_{ij}) *whose determinant has a given value* $C \neq 0$.

Proof. Consider the matrix of transformation (c_{ij}). Assign to the third column rational values not all zero. By equation (2) of the lemma the values of X_{i3} are given by

$$(5) \qquad X_{i3} = a_{i1}c_{13} + a_{i2}c_{23} + a_{i3}c_{33}, \qquad i = 1, 2, 3.$$

At least one of the X's is different from zero. For if all were zero, then a necessary and sufficient condition that equations (5) have a nontrivial solution in c_{13}, c_{23}, and c_{33} is that the determinant d of the coefficients be zero. This is contrary to hypothesis.

Since the cross-products in (4) must vanish, $b_{23}=0$; and by the lemma we have

$$X_{13}c_{12} + X_{23}c_{22} + X_{33}c_{32} = 0, \tag{6}$$

where not all X_{i3} ($i=1, 2, 3$) are zero as was shown above.

Equation (6) can be solved for c_{12}, c_{22}, c_{32} to give an infinite number of nontrivial rational solutions. We choose one solution such that $\phi(C_{11}, C_{21}, C_{31}) \neq 0$. So far the second and third columns of (c_{ij}) are determined. The same reasoning involved in showing that at least one of the X_{i3} is different from zero can be used to show that at least one of the X_{i2} is different from zero too.

To determine the elements of the first column we use the fact that $b_{12}=b_{13}=0$ and that the determinant of (c_{ij}) has a given value $C\neq 0$. To this effect we solve the following three equations for c_{11}, c_{21}, and c_{31}.

$$\begin{aligned} C_{11}c_{11} + C_{21}c_{21} + C_{31}c_{31} &= C \\ X_{12}c_{11} + X_{22}c_{21} + X_{32}c_{31} &= 0 \\ X_{13}c_{11} + X_{23}c_{21} + X_{33}c_{31} &= 0. \end{aligned} \tag{7}$$

A necessary and sufficient condition that these three equations will have solutions in c_{11}, c_{21}, and c_{31} not all zero is that $D\neq 0$, where D is the determinant of the coefficients in (7). But it can be shown that $D=\phi(C_{11}, C_{21}, C_{31})$ which is different from zero.

None of the b's in (4) is zero because the determinant of g is equal to $b_{11}b_{22}b_{23}$ and by the lemma it is equal to $C^2d\neq 0$.

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2. B. W. Jones, The arithmetic theory of quadratic forms, Carus Mathematical Monograph No. 10, Math. Assoc. of Am., 1950.

AN EXTENSION OF "AN APPROXIMATION FOR ANY POSITIVE INTEGRAL ROOT"

CAPT. EUGENE M. ROMER, USAF

Consider a recursion sequence of the form suggested by Taylor [1] to solve the problem $L=A^{1/s}$ using a desk calculator which can take square roots but not higher roots:

$$A_{n+1} = \gamma A_n + \beta(A_n^h A)^{1/(s+h)}, \tag{1}$$

where γ and β are constants to be chosen so as to make the convergence as rapid as possible. One purpose of this note is to show that more favorable selections of γ and β can be made than those considered by Taylor. We will define the frac-sary and sufficient condition that equations (5) have a nontrivial solution in c_{13}, c_{23}, and c_{33} is that the determinant d of the coefficients be zero. This is contrary to hypothesis.

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where γ and β are constants to be chosen so as to make the convergence as rapid as possible. One purpose of this note is to show that more favorable selections of γ and β can be made than those considered by Taylor. We will define the fractional error, ϵ_n, of the nth term by the equation

$$A_n = L(1 + \epsilon_n), \tag{2}$$

and noting that $L^s = A$, using (2) in (1) we obtain

$$1 + \epsilon_{n+1} = \gamma(1 + \epsilon_n) + \beta(1 + \epsilon_n)^{h/(s+h)}, \tag{3}$$

where $s+h = 2^k$ and h and k are integers.

The requirement that $\epsilon_{n+1} = 0$, where $\epsilon_n = 0$, yields

$$\gamma + \beta = 1. \tag{4}$$

We also note that $\epsilon_{n+1} = -1$ when $\epsilon_n = -1$; i.e., an initial estimate greater than zero is always required. The slope of ϵ_{n+1} *vs.* ϵ_n is given by

$$\frac{\partial \epsilon_{n+1}}{\partial \epsilon_n} = \gamma + \frac{\beta h}{s+h}(1 + \epsilon_n)^{-s/(s+h)}. \tag{5}$$

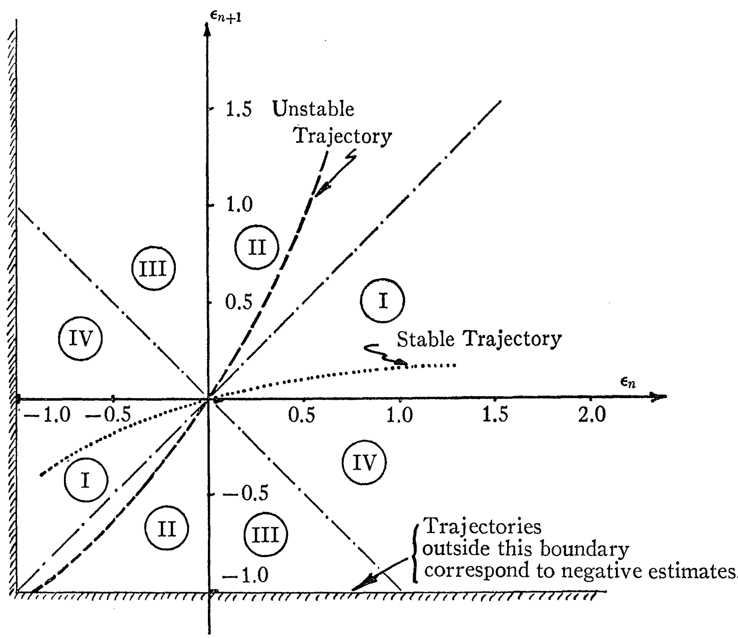

FIG. 1.

A sketch of some possible trajectories which could be realized from equation (3) is shown in Figure 1. The octants labeled I, II, III, IV correspond to differing classes of trajectories as follows:

Octant	*Remarks*
I	Stable trajectory; $\lvert\epsilon_{n+1}\rvert < \lvert\epsilon_n\rvert$; ϵ_{n+1} and ϵ_n have the same sign.
II	Conditionally stable trajectory; $\lvert\epsilon_{n+1}\rvert < \lvert\epsilon_n\rvert$; ϵ_{n+1} changes sign.
III	Conditionally unstable trajectory; $\lvert\epsilon_{n+1}\rvert > \lvert\epsilon_n\rvert$; ϵ_{n+1} changes sign.
IV	Unstable trajectory; $\lvert\epsilon_{n+1}\rvert > \lvert\epsilon_n\rvert$; no change in sign.

For convergence to the point, $\epsilon_{n+1}=\epsilon_n=0$, it is sufficient to require that in the vicinity of the origin

$$\left|\frac{\partial \epsilon_{n+1}}{\partial \epsilon_n}\right| < 1.$$

When this condition is used with (5), we obtain

$$\left|\gamma + \frac{\beta h}{s+h}\right| < 1, \tag{6}$$

or using (4) in (6):

$$\left|1 - \beta\frac{s}{s+h}\right| < 1 \quad \text{or} \tag{6a}$$

$$|\gamma s + h| < s + h. \tag{6b}$$

The most rapidly converging trajectories in the vicinity of the origin will be those whose slope is 0 at $\epsilon_n=0$; i.e. those whose trajectories satisfy the equation [from (5)]

$$\gamma + \frac{\beta h}{s+h} = 0. \tag{7}$$

The solution of (4) and (7) gives:

$$\beta = 1 + \frac{h}{s} \quad \text{and} \tag{8a}$$

$$\gamma = -\frac{h}{s}\cdot \tag{8b}$$

Using (8a) and (8b) in (1) we obtain:

$$A_{n+1} = \left[-\frac{h}{s}\right]A_n + \left[1 + \frac{h}{s}\right](A_n^h A)^{1/(s+h)}, \tag{9}$$

or

$$A_{n+1} = A_n\left[\left(1+\frac{h}{s}\right)(A_n^{-s}A)^{1/(s+h)} - \frac{h}{s}\right]. \tag{10}$$

Equation (10) is equation (5) of [2] recast into a slightly different form, which gives a trajectory shown in Figure 2. The range of ϵ_n for which (10) converges is $-1<\epsilon_n<\epsilon^*$ where ϵ_n^* is given by

$$\epsilon_n{}^* = \left(1+\frac{s}{h}\right)^{(1+(h/s))} - 1. \tag{11}$$

Note that as long as $0<h<s$, then

$$\left(1+\frac{s}{h}\right)^{(1+(h/s))} - 1 > \left(1+\frac{s}{h}\right) - 1 = \frac{s}{h};$$

i.e., $\epsilon_n^*>s/h$, (11a). Note also that (11) and (11a) give a much less restrictive range than would have been obtained by appealing to a Maclaurin expansion

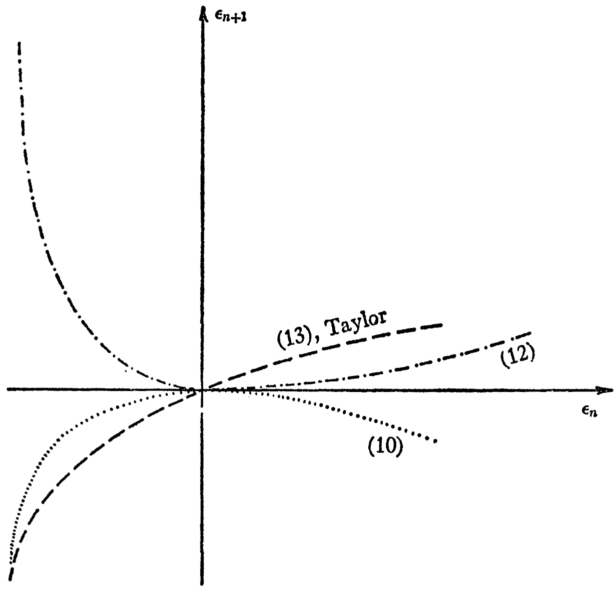

FIG. 2. Typical trajectories obtained from equations in text.

of (3) which would give $|\epsilon_n|<1$ as the allowable range for convergence. In the example below, i.e., the computation of the seventh root of 128, the range of convergence guaranteed by a Maclaurin expansion suggests an initial estimate, A_0, in the range $0<A_0<4$, whereas the range suggested by (11) and (11a) is $0<A_0<16$. In this application, however, the discussion of range of convergence is purely academic insofar as practical application is concerned, since the divergence is recognized by the appearance of a value of A_{n+1} in the range $A_{n+1}\leqq 0$ and can be eliminated by the simple expedient of taking the value of A_n and reducing it by some amount to use as the subsequent estimate, A_{n+1}. The appearance of a positive subsequent estimate generates a solution.

An alternate form can be obtained from (9) and (10) by substituting $(h-s)/2$ for h and leaving s unchanged:

$$A_{n+1} = A_n\left(\frac{s-h}{2s}\right) + \left(\frac{s+h}{2s}\right)\left(A_n^{(h-s)/2}A\right)^{2/(s+h)} \tag{12}$$

or

$$A_{n+1} = \frac{A_n}{2}\left[\left(\frac{s-h}{s}\right) + \left(\frac{s+h}{s}\right)\left(A_n^{-s}A\right)^{2/(s+h)}\right],$$

which is equation (4) of [2] in a slightly different form, and which gives a trajectory shown in Figure 2. Equation (12) will *always* converge on the correct answer for $0<A_n<\infty$ (as does any choice of γ and β which satisfies (6a), e.g., Taylor's final formula)

$$A_{n+1} = (A_n^h A)^{1/(s+h)} \tag{13}$$

subject to the restriction, $0<h<s$.

The choice of using (10) or (12) is arbitrary since both will converge to the correct answer and both exhibit increasing rapidity of convergence as the correct value is approached. An arbitrary criterion may be established by examining $\partial\epsilon_{n+1}/\partial\epsilon_n$ for both (10) and (12) for $\epsilon_n \gg 1$:

$$\text{for (10), } \frac{\partial\epsilon_{n+1}}{\partial\epsilon_n} \cong -\frac{h}{s} \tag{14a}$$

$$\text{for (12), } \frac{\partial\epsilon_{n+1}}{\partial\epsilon_n} \cong \frac{1}{2} - \frac{h}{2s}. \tag{14b}$$

From (14a, b), it would be reasonable to use (10) if $h/s \approx 0$, and use (12) if $h/s \approx 1$. Since, at $h/s = 1/3$, the magnitude of the slopes are equal, this is taken as the dividing value.

The form suggested by Taylor, i.e., equation (1), can be generalized to:

$$A_{n+1} = \gamma A_n + \beta(A_n^r A)^{2m/(s+h)}, \tag{15}$$

where m is an integer and r is to be determined. An equation similar to (3) can be obtained by requiring that the resulting error equation be independent of L or A; i.e., we require that

$$\left(\frac{r+s}{s+h}\right)2^m = 1 \tag{16}$$

or, since $s+h=2^k$, that

$$r = 2^{k-m} - s. \tag{17}$$

From (17), r will be an integer as long as $m \leqq k$. The error equations become

$$1 + \epsilon_{n+1} = \gamma(1+\epsilon_n) + \beta(1+\epsilon_n)^{2^m r/(s+h)} \tag{18}$$

and

$$\frac{\partial\epsilon_{n+1}}{\partial\epsilon_n} = \gamma + \beta\frac{2^m r}{s+h}(1+\epsilon_n)^{-(2^{m-k})s}. \tag{19}$$

Imposing the condition that the slope be zero at the origin gives us $\gamma + \beta 2^{m-k} r = 0$ or, using (4) and (17),

$$\beta = \frac{1}{2^m}\left(1+\frac{h}{s}\right) \quad \text{and} \quad \gamma = 1 - \frac{1}{2^m}\left(1+\frac{h}{s}\right).$$

(Note that for $m=0$, this gives rise to equation (10); for $m=1$, we obtain equation (12).) Equations of the form (15) will *always* be convergent as long as $\partial\epsilon_{n+1}/\partial\epsilon_n$ for large ϵ_n, is less than or equal to 1. From (19), $\gamma \leqq 1$ or

$$\frac{1}{2^m}\left(1+\frac{h}{s}\right) \geqq 0,$$

which imposes no real restriction. Note, however, that for large $\epsilon_n, \partial\epsilon_{n+1}/\partial\epsilon_n \approx \gamma$ or

$$\left.\frac{\partial\epsilon_{n+1}}{\partial\epsilon_n}\right|_{\epsilon_n \text{ large}} \approx 1 - \frac{2^{k-m}}{s}.$$

Consequently, the larger the value of m, the slower the rate of convergence, especially at large values of ϵ_n. On the other hand, increasing the value of m reduces the effort required to obtain subsequent estimates. In particular, if $m=k-1$, only one square root in each cycle must be computed.

For the problem considered by Taylor in [1], (that of finding the seventh root of 128, which is 2), a tabular comparison is shown below of the results using equations (10), (13), and (15), (for $m=k-1$).

TABLE 1a

Equation No.	m	s	h	r	γ	β
(13)	0	7	1	1	0	1
(10)	0	7	1	1	$-1/7$	$8/7$
(15)	2	7	1	-5	$5/7$	$2/7$

TABLE 1b. COMPUTED RESULTS

	Equation (13) *Taylor* (Reproduced from [1])	Equation (10)	Equation (15)
A_0	1.	1.	1.
A_1	1.834008085	1.944655161	3.96
A_2	1.978456025	1.990665203	2.92
A_3	1.997294225	1.999997348	2.31
A_4	1.999661584		2.05
A_5	1.999957694		2.00220
A_6	1.999994711		2.00000297

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1. G. R. Taylor, An approximation for any positive integral root, this MAGAZINE, 35 (1962) 107–108.

2. E. M. Romer, An iterative procedure for obtaining fractional roots of real numbers, Wright Air Development Center, TN 59–116, Wright-Patterson AF Base, Ohio, April, 1959.

A FORMULA CONCERNING TWIN PRIMES

GEORGETTA J. KOSTIS AND ROBERT L. PAGE, Nasson College

The sieve of Eratosthenes and the consequent formula $\pi(x)$ for the number of primes not exceeding x are well known from number theory. This paper is concerned with the application of these methods to twin primes.

Listing the primes up to a certain limit, we eliminate all primes which are not the smaller of a pair of twin primes. Therefore, we first cross out the prime 2. No prime of the form $3n+1$ is the smaller of a pair of twin primes since we obtain the composite number $3n+3$ when we add 2. Hence, we cross out all primes of the form $3n+1$. We similarly eliminate primes of the form $5n+3$, $7n+5$, $11n+9$, etc., where $n=1, 2, 3, \cdots$.

~~2~~ 3 5 ~~7~~ 11 ~~13~~ 17 ~~19~~ ~~23~~ 29 ~~31~~ ~~37~~.

Each of the remaining primes is the smaller of a pair of twin primes and we merely add 2 in order to obtain the other member.

We now develop a formula for the function $\tau(x)$, the number of pairs of twin primes whose smaller members do not exceed x. First, we subtract from $\pi(x)$ the number of integers of the form $3n+1$ which are >1 and which do not exceed x, namely $[(x-1)/3]$. Since this number includes some composite numbers, however, we add the number of integers of the form $3n+1$ which are divisible by 2 and which do not exceed x, $[(x+2)/6]$, and the number of integers of the form $3n+1$ which are divisible by 5 and which do not exceed x, $[(x+5)/15]$. In order to avoid double counting, we now subtract the number of integers of the form $3n+1$ which are divisible by 10 and which do not exceed x, namely $[(x+20)/30]$.

Similarly, the next four terms of the formula concern integers of the form $5n+3$ which are >3. In the next term we again avoid double counting by adding the number of integers having both of the forms $3n+1$ and $5n+3$, $[(x+2)/15]$. Since some of these are composite, we now subtract the number of integers having the forms $3n+1$ and $5n+3$ and divisible by 2, $[(x+2)/30]$. Finally, we subtract 1 to account for the prime 2 and obtain:

$$\tau(x) = \pi(x) - \left[\frac{x-1}{3}\right] + \left[\frac{x+2}{6}\right] + \left[\frac{x+5}{15}\right] - \left[\frac{x+20}{30}\right] - \left[\frac{x-3}{5}\right] + \left[\frac{x+2}{10}\right] + \left[\frac{x-3}{15}\right] - \left[\frac{x+12}{30}\right] + \left[\frac{x+2}{15}\right] - \left[\frac{x+2}{30}\right] - 1$$

for all integers $x<7^2-2=47$. For example, $\tau(28)=9-9+5+2-1-5+3+1-1+2-1-1=4$ while $\tau(29)=10-9+5+2-1-5+3+1-1+2-1-1=5$.

As an illustration of the way in which the formula operates, let us observe the effect of each term in square brackets when calculating $\tau(28)$ (see Table I).

TABLE I

Form	Operation	~~2~~ 3 5 ~~7~~ 11 ~~13~~ 17 ~~19~~ ~~23~~
$3n+1, >1$	Sub.	4 7 10 13 16 19 22 25 28
$3n+1$, div. by 2	Add	4 10 16 22 28
$3n+1$, div. by 5	Add	10 25
$3n+1$, div. by 10	Sub.	10
$5n+3, >3$	Sub.	8 13 18 23 28
$5n+3$, div. by 2	Add	8 18 28
$5n+3$, div. by 3	Add	18
$5n+3$, div. by 6	Sub.	18
$3n+1$ and $5n+3$	Add	13 28
$3n+1$, $5n+3$ div. by 2	Sub.	28
The prime 2	Sub.	2

The reason that the formula is valid only for $x<47$ rather than $7^2=49$ is that 47 is a prime of the form $7n+5$ which is not eliminated by any terms of the formula. However, any prime P smaller than 47 which is of the form $7n+5$ and which is not the smaller of a pair of twin primes will yield a composite number C upon adding 2. Now C has a prime factor $p<7$ and $P=C-2=pm+p-2$. Therefore, P will have already been accounted for by the term for numbers of the form $pm+p-2$. For example, $19=7(2)+5=3(6)+1$ is eliminated by the term for numbers of the form $3m+1$.

The formula for $\tau(x)$ may be extended to larger values of x by including terms for the subsequent primes. It becomes complicated very rapidly, however, and the formula for $x<(11)^2-2=119$, obtained by considering the additional prime 7, contains 40 terms.

The results of this paper are contained in a Senior Thesis submitted at Nasson College by Miss Kostis.

SUBADDITIVITY IS A ROTATION INVARIANT

RICHARD G. LAATSCH, Miami University, Ohio

A real-valued function f defined on a set H of real numbers is *subadditive* on H if $f(x+y) \leqq f(x)+f(y)$ for all $x, y \in H$ such that $x+y \in H$. A thorough introduction to the behavior of subadditive functions has been written by Hille and Phillips [1]. In this theory the function f°, defined by $f^\circ(x)=f(x)/x$ for positive x, plays a prominent part. One result involving f° is the following: If f is defined on $(0, \infty)$ and f° is nonincreasing there, then f is subadditive on $(0, \infty)$. Bruckner [2] also has obtained some results involving f°.

The function given by $|\sin x|$ is readily shown to be subadditive by using the addition formula, and the analogue of a theorem of Bruckner [2] readily establishes that, if the graph of this function is rotated through an angle less than $\pi/4$, then the resulting function is also subadditive. Ideas such as these have motivated the following general result, which has application to continuous subadditive functions.

THEOREM. *Let f be subadditive on the set of all real numbers and let θ, $0<\theta<\pi/2$ (or $-\pi/2<\theta<0$), be a rotation of axes such that the graph $v=f(u)$ in the rectangular cartesian (u, v)-system is the graph of a (single-valued) real function for all rotations α of the axes such that $0 \leqq \alpha \leqq \theta$ (respectively, $\theta \leqq \alpha \leqq 0$). Then the function g obtained by referring the graph of f to the (u', v')-system resulting from the rotation θ is subadditive on the set of all real numbers.*

Proof. It will be shown that, if g is not subadditive, there exist points of the graph which determine a line perpendicular to a rotated position of the u-axis for some rotation α between 0 and θ.

Let x', y', and z' be values of u' such that $x'+y'=z'$. Let x, y, and z be the u-coordinates of $(x', g(x'))$, $(y', g(y'))$, and $(z', g(z'))$, respectively. Suppose that $g(z')>g(x')+g(y')$, and let $G=g(z')-g(x')-g(y')$. From the rotation formula $f(u)=u'(\sin\theta)+g(u')(\cos\theta)$ it follows that

$$f(z)-f(x)-f(y)=(z'-x'-y')(\sin\theta)+G(\cos\theta)=G(\cos\theta). \tag{1}$$

In the same way, the formula $u=u'(\cos\theta)-g(u')(\sin\theta)$ yields

$$x+y-z=G(\sin\theta). \tag{2}$$

If $\theta>0$, equation (2) yields $x+y=z+G(\sin\theta)>z$, which implies that the point $(x+y, f(x+y))$ is to the right of the line $u=z$. Using equation (1) and the subadditivity of f,

$$f(x+y) \leqq f(x)+f(y)=f(z)-G(\cos\theta), \tag{3}$$

which implies that $(x+y, f(x+y))$ is on or below the line $u'=z'$. Therefore, the points $(z, f(z))$ and $(x+y, f(x+y))$ determine a line which makes an angle α, $0<\alpha\leqq\theta$, with the vertical line $u=z$. For this rotation α of the (u, v)-axes these two distinct points have the same abscissa.

If $\theta<0$, $\sin\theta<0$, so that equation (2) implies $x+y<z$. The inequality (3) still holds, which means that the rotation again exists for which $(z, f(z))$ and $(x+y, f(x+y))$ have the same abscissa.

Although the property $\cos\theta > 0$ was used in the proof, the restriction $-\pi/2 < \theta < \pi/2$ can be removed by using the sum of rotations, each smaller than $\pi/2$. Of course, the theorem can also be applied to some other cases, such as that of f defined on $[0, \infty)$ with $f(0) = 0$.

References

1. Einar Hille and R. S. Phillips, Functional analysis and semigroups, Amer. Math. Soc. Colloq. Publ., 31 (1957).

2. Andrew Bruckner, Minimal superadditive extensions of superadditive functions, Pacific J. Math., 10 (1960) 1155–1162.

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted. Solutions should be submitted on separate, signed sheets. Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles 29, California.

PROPOSALS

551. *Proposed by C. W. Trigg, San Diego, California.*

Reconstruct the following multiplication, given that $R = 3$.

$$\begin{array}{ccccccccccccc}
 & & & & & & & * & * & * & * & * & R \\
 & & & & & & R & * & * & * & * & * & * \\
\hline
 & & & & & & R & * & R & * & R & * & C \\
 & & & & & & * & * & * & * & * & R & \\
 & & & & * & * & * & * & * & * & Y & & \\
 & & & & * & * & * & R & * & S & & & \\
 & & * & * & * & * & * & * & T & & & & \\
 & * & * & * & * & * & * & A & & & & & \\
C & R & Y & S & T & A & L & & & & & & \\
\hline
* & * & * & * & * & * & * & * & * & * & * & * & *
\end{array}$$

552. *Proposed by Yasser Dakkah, S. S. Boys School, Qalqilya, Jordan.*

Given a right circular cylinder with base of radius r, height h, volume V, and lateral area A. If $r + h = k$, a constant, prove:

$$1.\ V \leqq \frac{4\pi k^3}{27} \qquad 2.\ A \leqq \tfrac{1}{2}\pi k^2.$$

Although the property $\cos\theta > 0$ was used in the proof, the restriction $-\pi/2 < \theta < \pi/2$ can be removed by using the sum of rotations, each smaller than $\pi/2$. Of course, the theorem can also be applied to some other cases, such as that of f defined on $[0, \infty)$ with $f(0)=0$.

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1. Einar Hille and R. S. Phillips, Functional analysis and semigroups, Amer. Math. Soc. Colloq. Publ., 31 (1957).
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$$\begin{array}{ccccccccccccc}
 & & & & & & & \ast & \ast & \ast & \ast & \ast & R \\
 & & & & & & R & \ast & \ast & \ast & \ast & \ast & \ast \\
\hline
 & & & & & & R & \ast & R & \ast & R & \ast & C \\
 & & & & & & \ast & \ast & \ast & \ast & \ast & R & \\
 & & & & \ast & \ast & \ast & \ast & \ast & \ast & Y & & \\
 & & & & \ast & \ast & \ast & R & \ast & S & & & \\
 & & \ast & \ast & \ast & \ast & \ast & \ast & T & & & & \\
 & \ast & \ast & \ast & \ast & \ast & \ast & A & & & & & \\
C & R & Y & S & T & A & L & & & & & & \\
\hline
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast
\end{array}$$

552. *Proposed by Yasser Dakkah, S. S. Boys School, Qalqilya, Jordan.*

Given a right circular cylinder with base of radius r, height h, volume V, and lateral area A. If $r+h=k$, a constant, prove:

$$1.\ V \leqq \frac{4\pi k^3}{27} \qquad 2.\ A \leqq \tfrac{1}{2}\pi k^2.$$

553. *Proposed by Daniel I. A. Cohen, Princeton University.*

Prove that the determinant of the n by n magic square formed from the numbers 1 to n^2 is divisible by n if n is odd and by n^2+1 if n is even.

554. *Proposed by Joseph Verdina, Long Beach State College, California.*

Find the asymptote to the curve given by

$$y^3 - x^3 + ax^2 = 0.$$

555. *Proposed by Kaidy Tan, Fukien Normal College, Foochow, Fukien, China.*

With O as any point on the median AM of a triangle ABC, produce BO and CO meeting AC and AB at E and F, respectively. If $AB>AC$, prove that $BE>CF$.

556. *Proposed by Sidney Kravitz, Dover, New Jersey.*

Professor Adams wrote on the blackboard a polynomial, $f(x)$, with integer coefficients and said, "Today is my son's birthday, and when we substitute x equal to his age, A, then $f(A)=A$. You will also note that $f(0)=P$ and that P is a prime number greater than A." How old is Professor Adams' son?

557. *Proposed by Roy Feinman, Rutgers University.*

Let A, B, and C be the vertex angles of any triangle. There exists an identity $F(\cos A, \cos B, \cos C)\equiv 0$ which is symmetric in $\cos A$, $\cos B$, and $\cos C$, and is of degree 2 in each of these. What is it?

SOLUTIONS

Late Solutions

511. *Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania.*

516. *Dee Fuller, University of Georgia.*

518. *David M. Levin, Lowell High School, San Francisco, California; Dee Fuller, University of Georgia.*

519. *Josef Andersson, Vaxholm, Sweden.*

523. *Josef Andersson, Vaxholm, Sweden; Daniel I. A. Cohen, Princeton University.*

526. *Daniel I. A. Cohen, Princeton University.*

528. *Josef Andersson, Vaxholm, Sweden; Daniel I. A. Cohen, Princeton University.*

524, 529. *Josef Andersson, Vaxholm, Sweden.*

Colored Square

526. [September 1963]. *Proposed by C. W. Trigg, Los Angeles City College.*

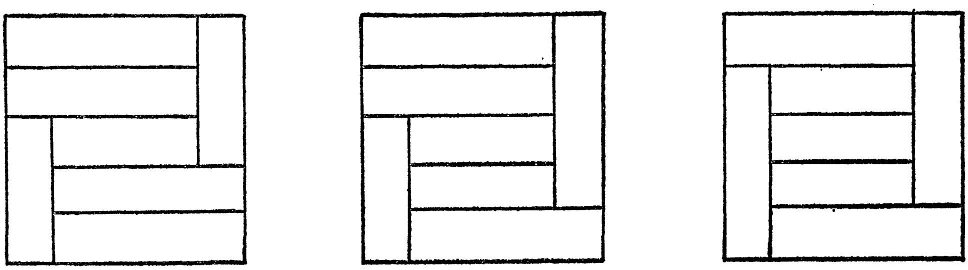
Each of four strips, 4″ by 1″, and three strips, 3″ by 1″, has a different color of the spectrum. Into how many distinct square designs may they be arranged?

Solution by Alan Sutcliffe, Knottingley, Yorkshire, England.

First, consider the number of ways that the strips can be arranged regardless of color, and also neglecting rotation and reflection. Taking those cases where there are more strips placed horizontally than vertically, clearly there cannot be more than five strips placed horizontally. If there were four strips horizontally and three vertically, then the three vertical ones would overlap to leave only two squares vacant in row 3. Hence, the only arrangements that are possible will have five strips horizontally and two vertically.

$A1$	$B1$	$C1$	$D1$	$E1$
$A2$	$B2$	$C2$	$D2$	$E2$
$A3$	$B3$	$C3$	$D3$	$E3$
$A4$	$B4$	$C4$	$D4$	$E4$
$A5$	$B5$	$C5$	$D5$	$E5$

FIG. 1.



(a) (b) (c)

FIG. 2.

The two vertical strips must be in columns A and E, for otherwise there would be one or two squares left in row 3 which could not be covered by a horizontal strip. The two vertical strips must also cover opposite corners, say $A5$ and $E1$, for otherwise either row 1 or row 5 would be empty and could not be covered entirely by a horizontal strip. This leaves 3 possibilities: both vertical strips are 3″ ones; one is 3″ and the other 4″; both are 4″. With these in place the positions of the other strips are determined, and are as shown in Figure 2, except for rotation of the 3″ strips in (c).

Consider next the variants of these arrangements by rotation and reflection. It is easily seen that there are 4 versions of (a), 8 versions of (b), and 4 versions of (c).

Lastly, consider the ways in which each of these arrangements can be made with the colored strips. In each case there are 6 ways in which the 3″ strips can be arranged, and 24 ways in which the 4″ strips can be arranged; hence, the total number of arrangements of the seven colored strips is $6\cdot 24(4+8+4)=2{,}304$.

To find the number when arrangements which differ only by rotation are not counted as distinct, note that each arrangement has just four forms which differ only by rotation. (The basic arrangements (a) and (c) have only two forms each which differ by rotation, but the arrangements of colored rectangles based on them have four forms each.) Hence, the number of arrangements, when those that differ only by rotation are not counted as distinct, is 576.

Also solved by the proposer. Two incorrect solutions were received.

Four Generations

530. [November 1963]. *Proposed by Maxey Brooke, Sweeny, Texas.*

"Here is something interesting," said Bobby's father. "The product of my age and Bobby's age remains the same even when the two digits in each age are reversed. And our ages are not divisible by 11."

"That's nothing," said Bobby's grandfather. "The product of my age and Bobby's also remains the same when the digits in each age are reversed."

"You've got nothing on me," said Bobby's great-grandfather. "The product of my age and Bobby's age also remains the same when the two digits in each age are reversed."

How old is Bobby?

Solution by Monte Dernham, San Francisco, California.

If $10v+x$ denotes Bobby's age, and $10t+u$ the age of any of the three gentlemen present at the confab, then

$$(10v + x)(10t + u) = (10x + v)(10u + t),$$

whence $x=vt/u$.

There are exactly three solutions.

Suppose Bobby is not yet 20, then $v=1$ and $x=t/u$, restricted to

$$x = 2,\quad 3,\quad 4,\quad 5,\quad 6,\quad 7,\quad 8,\quad \text{or}\quad 9.$$

$$\text{and } t/u = \begin{cases} 2/1 & 3/1 & 4/1 & 5/1 & 6/1 & 7/1 & 8/1 & \text{or} & 9/1. \\ 4/2 & 6/2 & 8/2 \\ 6/3 & 9/3 \\ 8/4 \end{cases}$$

Now, in order to make provision for father, for granddaddy, and for great-grandpa as well, it is at once apparent that all $x>3$ must be rejected. Thus Bobby may be 12, in which case his dad, who must then be over 21, would be 42, grandpa 63, and great-grandfather 84 (first column). A remaining possibility: Bobby 13, father 31, granddad 62, and great-granddaddy 93 (second column).

On the other hand, if Bob is already in his 20's, $x=2t/u$, and pursuing the same method, we find he must then be 24, while the three paternal ancestors of his here engaged in conversation are, as before, 42, 63, and 84, respectively.

The foregoing results are easily verified, and any supposition Bob may be still older is readily discovered to be untenable.

Also partially solved by Albert Adell, Flushing, New York; Merrill Barneby, University of North Dakota; A. R. Billimoria, New York, New York; Sister Marie Blanche, The Immaculata, Washington, D. C.; Joseph B. Bohac, St. Louis, Missouri; Dermott A. Breault, Sylvania Electronics Systems, Waltham, Massachusetts; Allan W. Brunson, Euclid, Ohio; Daniel I. A. Cohen, Princeton University; Sister M. Coleman, Marywood College, Pennsylvania; Anton Glasser, Pennsylvania State University, Ogontz Campus; J. A. H. Hunter, Toronto, Ontario, Canada; J. Lawrence Katz, Rensselaer Polytechnic Institute; John W. Milsom, Texas A & I College; Wahin Ng, San Francisco, California; Kenneth Siler, St. Mary's College, California; Alan Sutcliffe, Knottingley, Yorkshire, England (complete solution); W. R. Talbot, Morgan State College, Maryland; C. W. Trigg, San Diego, California; Ralph N. Vawter, St. Mary's College, California; Dale Woods, Northeast Missouri State Teachers College; and the proposer. Late solution by Josef Andersson, Vaxholm, Sweden.

Other ages were possible if restrictions on the time between generations were relaxed or if zero could be used as a first integer in the integer xy.

A Separable D. E.

531. [November 1963]. *Proposed by Ben B. Bowen, Vallejo Junior College, California.*

Given the differential equation

$$\sqrt{\{1-y^2\}}\,dx = \sqrt{\{1-x^2\}}\,dy,$$

a student immediately wrote the solution

$$x\sqrt{\{1-y^2\}} = y\sqrt{\{1-x^2\}} + c.$$

His text book gave the solution $\sin^{-1} x - \sin^{-1} y = c$.

Given the differential equation $f(y)dx = f(x)dy$, find f such that $xf(y) - yf(x) = c$ is a solution.

Solution by the proposer.

Equating dy/dx taken from the given equation and solution, we have:

$$\frac{yf'(x)}{xf'(y)} = \frac{f(y)}{f(x)}$$

or, separating variables,

$$\frac{f(x)f'(x)}{x} = \frac{f(y)f'(y)}{y}$$

and if $x \neq y$, this will be true if

$$\frac{f(t)f'(t)}{t} = c \quad (c \text{ a positive or negative real number}).$$

Upon integrating, we find that f, real, can take on the forms:

$$f(t) = \pm \sqrt{\{a^2 + b^2t^2\}},$$
$$f(t) = \pm \sqrt{\{a^2 - b^2t^2\}}, \qquad |bt| \leqq a$$
$$f(t) = \pm \sqrt{\{b^2t^2 - a^2\}}, \qquad |bt| \geqq a.$$

Upon separating the variables in the given D.E., we find that these are just the functions that integrate into arcsin, arcsinh, and arccosh.

Also solved by A. R. Billimoria, New York, New York; Martin J. Cohen, Beverly Hills, California; Roop N. Kesarwani, Wayne State University, Michigan; M. S. Klamkin, SUNY at Buffalo, New York; and Lewis Bayard Robinson, Baltimore, Maryland. Late solution by Josef Andersson, Vaxholm, Sweden.

Parabola Construction

532. [November 1963]. *Proposed by Josef Andersson, Vaxholm, Sweden.*

The direction of the axis of a parabola is given. Construct the parabola if three points on the curve are also given.

Solution by W. R. Talbot, Morgan State College, Maryland.

No parabola is possible if the three points lie on a line which meets the axis finitely, and if two or more of the points lie on a line parallel to the axis, the parabola consists of two distinct or coincident lines parallel to the axis. Let the given points C, D, E be noncollinear and no two of them on a line parallel to the axis.

Inasmuch as a parabola is tangent to the line at infinity in the direction of the axis, we may regard that point on the axis as the two points A and B moved into coincidence. Now that five points on the conic are known it may be constructed by use of Pascal's Line.

Let lines be formed and numbered as follows: AB, 1 (line at infinity); BC, 2 (parallel to axis); CD, 3; DE, 4; EF, 5 (where F is arbitrary except that EF is not parallel to 2). Lines 1 and 4 meet in P and 2 and 5 in Q. Then PQ, Pascal's Line, is a parallel to 4 through Q. Let it meet 3 in R which is joined to A by a parallel to the axis. AR meets EF in a new point on the parabola.

Also solved by Merrill Barneby, University of North Dakota; and the proposer.

Square Partitions

533. [November 1963]. *Proposed by David L. Silverman, Beverly Hills, California.*

What is the largest integer which cannot be partitioned into distinct squares?

Solution by J. A. H. Hunter, Toronto, Ontario, Canada.

By quick trial, it is seen that 128 cannot be represented as the sum of distinct squares.

Observing that $(1^2+2^2+ \cdots +10^2)=385$, whence a representation for N will lead to a representation for $385-N$, we now test the numbers from 129 up to 192, and find that all can be represented using no square greater than 10^2: this must then apply for numbers 193 to 249.

Adding $121=11^2$, to each such representation in the sequence 129 to 249, we obtain representations for the further sequence up to 370 without using $144=12^2$. We continue by adding $144=12^2$ to the previous representations of the 144 numbers 227 to 370, and then proceed similarly adding $169=13^2$, etc.: this being justified because thereafter we have $2m^2>(2m+1)^2$.

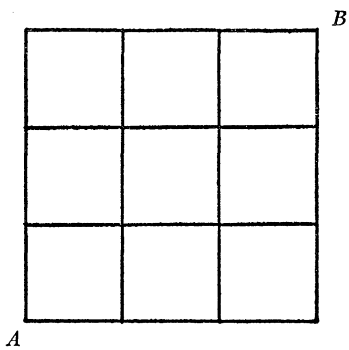
Hence, all integers greater than 128 can be represented as the sums of distinct squares, and 128 is the greatest integer that cannot be so represented.

Also solved by Alan Sutcliffe, Knottingley, Yorkshire, England; and the proposer.

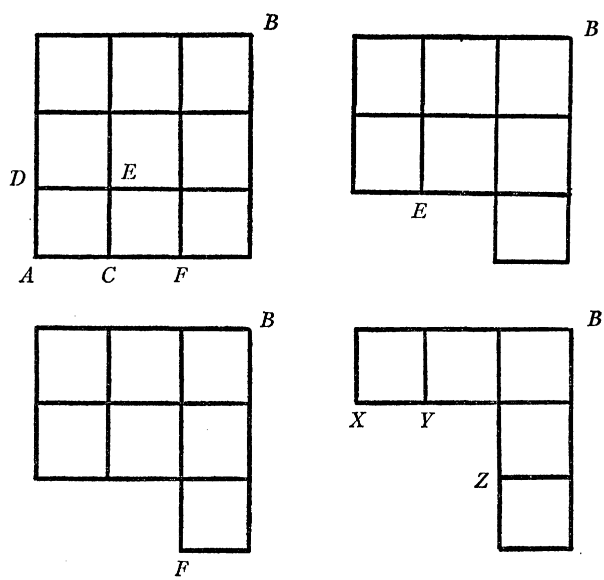
A–maze–ing

534. [November 1963]. *Proposed by Brother U. Alfred, St. Mary's College, California.*

Given a three-by-three grid as shown



Determine the number of paths from A to B under the limitation that no point of the grid is covered twice by any one path.



Solution by the proposer.

By symmetry it should be clear that the number of paths obtained by going to C should be the same as the number obtained by going to D. Hence we limit ourselves to the former.

The number of paths from C will equal the sum of the number of paths from F plus the number of paths from E. From E we may move in three possible directions to X, Y, or Z. It is not too difficult to show that the number of paths from X to B is 16, from Y to B is 12, and from Z to B is 12.

One can proceed similarly to break down the figure for F, the final result being 52 paths. Thus there are 40 plus 52 paths from C, and since there would be a like number from D, the total is 184. (See figure).

Also solved by Albert Adell, Flushing, New York; and Alan Sutcliffe, Knottingley, Yorkshire, England. Two incorrect solutions were received.

Graphical Solution of a D. E.

535. [November 1963]. *Proposed by Murray S. Klamkin, State University of New York at Buffalo.*

It is known that if the family of integral curves of the linear differential equation $y'+P(x)y=Q(x)$ is cut by the line $x=a$, then the tangents at the points of intersection are concurrent. Prove, conversely, that if for a first order equation $y'=F(x, y)$ the tangents (as above) are concurrent, then $F(x, y)$ is linear in y.

Solution by Roop N. Kesarwani, Wayne State University, Michigan.

Let the point of intersection with the line $x=a$ of a typical member of the family of integral curves of $y'=F(x, y)$ be (a, y_0). If a is fixed, y_0 clearly depends on the parameter of the family.

The tangent at the point of intersection to the integral curve is then $y-y_0=F(a, y_0)(x-a)$. All such tangents pass through the same point, say (A, B). Therefore $B-y_0=F(a, y_0)(A-a)$, or

$$F(a, y_0) = \frac{B - y_0}{A - a},$$

proving that $F(x, y)$ is linear in y.

Also solved by Martin J. Cohen, Beverly Hills, California; Wilbur H. McKenzie, City College of San Francisco; and the proposer. Late solution by Josef Andersson, Vaxholm, Sweden.

Klamkin pointed out that this result provides a basis for a graphical solution of the given differential equation. See H. Betz, P. B. Burcham, and G. M. Ewing, *Differential Equations with Applications*, 1954; and M. S. Klamkin, *On a Graphical Solution of the First Order Linear Differential Equation*, Amer. Math. Monthly, 61 (1954) 565–7.

Concurrent Lines

536. [November 1963]. *Proposed by D. Moody Bailey, Princeton, West Virginia.*

A line through the incenter I of triangle ABC meets sides BC, CA, and AB

at the respective points M, N, O. Points D, E, and F are chosen on sides BC, CA, and AB so that

$$\frac{BD}{DC} = -\frac{c}{b}\cdot\frac{BM}{MC}, \qquad \frac{CE}{EA} = -\frac{a}{c}\cdot\frac{CN}{NA},$$

and

$$\frac{AF}{FB} = -\frac{b}{a}\cdot\frac{AO}{OB}$$

where a, b, and c are the sides opposite vertices A, B, and C of the triangle. Show that rays AD, BE, and CF are concurrent at a point P which lies on the circumcircle of triangle ABC.

Solution by the proposer.

The theorem of Menelaus yields

$$\frac{BM}{MC}\cdot\frac{CN}{NA}\cdot\frac{AO}{OB} = -1.$$

Then

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = -\frac{c}{b}\cdot-\frac{a}{c}\cdot-\frac{b}{a}\cdot\frac{BM}{MC}\cdot\frac{CN}{NA}\cdot\frac{AO}{OB} = 1$$

and the converse of Ceva's theorem shows that rays AD, BE, and CF are concurrent.

Construct rays BI and CI to meet sides CA and AB at respective points E' and F'. We then know that

$$\frac{AF'}{F'B}\cdot\frac{BO}{OA} + \frac{AE'}{E'C}\cdot\frac{CN}{NA} = 1 \quad \text{or} \qquad (1) \qquad \frac{b}{a}\cdot\frac{BO}{OA} + \frac{c}{a}\cdot\frac{CN}{NA} = 1$$

for all positions of line MNO through incenter I. This result is discussed in the article entitled "A Triangle Theorem" found in the April, 1960, issue of *School Science and Mathematics.*

If P lies on the circumcircle,

$$\frac{BF}{FA}\cdot\frac{b^2}{a^2} + \frac{CE}{EA}\cdot\frac{c^2}{a^2} = -1,$$

and conversely.

This is shown in the solution of Problem No. 412 recorded in the January-February, 1961, issue of this MAGAZINE. Substituting the values of BF/FA and CE/EA listed in the problem, it is found that

$$\frac{BF}{FA}\cdot\frac{b^2}{a^2} + \frac{CE}{EA}\cdot\frac{c^2}{a^2} \quad \text{becomes} \quad -\frac{b}{a}\cdot\frac{BO}{CA} - \frac{c}{a}\cdot\frac{CN}{NA}$$

which is equal to negative unity when viewed in the light of (1). Since

$$\frac{BF}{FA}\cdot\frac{b^2}{a^2}+\frac{CE}{EA}\cdot\frac{c^2}{a^2}=-1,$$

point P must lie on the circumcircle of triangle ABC.

As variable line MNO rotates about the incenter I, point P will trace out the circumcircle of triangle ABC.

Also solved by Josef Andersson, Vaxholm, Sweden.

Comment on Quickie 306

Q306. [January 1963]. *Submitted by M. S. Klamkin.*

Comment by Larry Murrell, Arlington State College, Texas.

There seems to be an error in Q306 and its solution. We should have

$$\begin{aligned}
&\left(\frac{1}{11}-\frac{1}{1100}\right)+\left(\frac{1}{111}-\frac{1}{111000}\right)+\cdots\\
&=\frac{1}{11}\left(1-\frac{1}{100}\right)+\frac{1}{111}\left(1-\frac{1}{1000}\right)+\cdots\\
&=\frac{1}{11}\left(\frac{99}{100}\right)+\frac{1}{111}\left(\frac{999}{1000}\right)+\cdots\\
&=9\left[\frac{1}{100}+\frac{1}{1000}+\cdots\right]\\
&=\frac{9}{100}\left[1+\frac{1}{10}+\left(\frac{1}{10}\right)^2+\cdots\right]\\
&=\frac{9}{100}\left[\frac{1}{1-\frac{1}{10}}\right]=\frac{1}{10}
\end{aligned}$$

Therefore,

$$\frac{1}{11}+\frac{1}{111}+\frac{1}{1111}+\cdots=\frac{1}{10}+\frac{1}{1100}+\frac{1}{111000}+\cdots.$$

Comment on Problem 517

517. [May 1963 and January 1964]. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Comment by Josef Andersson, Vaxholm, Sweden.

This property of the areas characterizes in a way the parabola. If, in fact, $r=f(\phi)$ is the equation of a curve K in polar coordinates, associated to orthonormal coordinates $X'OX$, $Y'OY$ and that for each point P of K we construct PP' equivalent to $OQ(r, 0)$ the point P' traces the curve K'. Let A_1 and A_2

represent the areas between $X'OX$, OP, K and K, PP', K', $X'OX$ respectively. The condition

$$(1) \qquad \frac{1}{2}r^2 = \frac{dA_1}{d\phi} = \frac{1}{2}\cdot\frac{dA_2}{d\phi} = \frac{1}{2}\cdot\frac{dA_2}{d(r\sin\phi)}\cdot\frac{d(r\sin\phi)}{d\phi} = \frac{1}{2}r\frac{d(r\sin\phi)}{d\phi}$$

gives

$$\frac{d(r\sin\phi)}{r\sin\phi} + \frac{d\left(\cot\frac{\phi}{2}\right)}{\cot\frac{\phi}{2}} = 0, \qquad 2r\cos^2\frac{\phi}{2} = \text{constant}.$$

Therefore, K is a parabola with O as focus, $X'OX$ as axis and therefore K' is the directrix. The hypothesis and the ratio $\frac{1}{2}$ is deducted immediately from (1) if we take at first one of the points at the vertex.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q336. Show that the only factorization of homogeneous polynomials into polynomials is into homogeneous ones.
[*Submitted by M. S. Klamkin.*]

Q337. Demonstrate the relative-prime property of the Fermat numbers $F_k = 2^{2^k}+1$, $k=1, 2, 3, \cdots$ by establishing the identity mentally,

$$(2+1)(2^2+1)(2^{2^2}+1)(2^{2^3}+1)\cdots(2^{2^n}+1) \equiv (2^{2^{n+1}}+1) - 2.$$

[*Submitted by Dewey C. Duncan.*]

Q338. Sum the series

$$\sum_{n=1}^{\infty} \frac{n+1}{2^n}.$$

[*Submitted by David L. Silverman.*]

Q339. An array of 2 million points is completely enclosed by a circle having a diameter of one inch. Does there exist a straight line having exactly 1 million of these points on each side of the line? Why?
[*Submitted by Herbert Wills.*]

(Answers on page 176)

ANSWERS

A336. Proof for three variables.

Assume

$$H(x, y, z) = F(x, y, z)G(x, y, z).$$

But H can be expressed in the form

$$x^n P\left(\frac{y}{x}, \frac{z}{x}\right).$$

Let $r = y/x$ and $s = z/x$, then

$$x^n P(r, s) = F(x, rx, sx)G(x, rx, sx).$$

Now it follows that

$$F(x, rx, sx) = x^{n_1} P_1(r, s)$$

$$G(x, rx, sx) = x^{n_2} P_2(r, s).$$

Since the only factorizations of x^n are of the form $x^{n_1} \cdot x^{n_2}$, where $n_1 + n_2 = n$. Whence F and G are homogeneous.

A337. Multiply the factors beginning at the left, first multiplying by 1 in the form $2-1$, obtaining in succession 2^2-1, $2^{2^2}-1$, and finally $2^{2^{n-1}}-1$, or $2^{2^{n-1}}+1-2$. Whence any factor of F_{n+1} and F_h, $(h < n+1)$ must also divide 2. Since F_i is odd, there is no prime divisor common to F_{n+1} and F_h, $(h < n+1)$.

A338. The series

$$\sum_{n=1}^{\infty} \frac{n+1}{2^n} = 2(\tfrac{1}{2}) + 3(\tfrac{1}{2} \cdot \tfrac{1}{2}) + 4(\tfrac{1}{2} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}) \cdot \cdot \cdot$$

represents the expected number of children a couple must have before they have a child of each sex. Without loss of generality, let the first child be a boy. Since "half a girl" is "expected" at each succeeding birth, two additional births are expected in order to produce one girl. Thus, the expectation is three, to which the series must sum.

A339. Consider all of the lines determined by pairs of points in the array. Pick a new point belonging to none of these lines and outside the circle. Consider a line through this new point and to the left of the array of points. As this line is rotated about the new point and toward the right of the array, it passes exactly one point of the array at a time. Hence, rotate this line until it passes exactly 1 million of the points.

(Quickies on page 206)

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